Example 91. Determine the shape (but not the exact numbers involved) of the partial fraction decomposition of the following rational functions.

(a) $\dfrac{x^2-2}{x^4-x^2} = \dfrac{x^2-2}{x^2(x-1)(x+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-1} + \dfrac{D}{x+1}$

(b) $\dfrac{x^3-7x+1}{x^2(x^2+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx+D}{x^2+1}$

(c) $\dfrac{x^2+5}{(x+2)^3(x^2+1)^2} = \dfrac{A}{x+2} + \dfrac{B}{(x+2)^2} + \dfrac{C}{(x+2)^3} + \dfrac{Dx+E}{x^2+1} + \dfrac{Fx+G}{(x^2+1)^2}$

Example 92. Evaluate $\displaystyle\int \frac{x^7+3x+1}{x^4+x^2}\mathrm{d}x$.

Solution.

- First, note that the numerator degree is not less than the denominator degree. Hence, we have to do long division first: (review, if necessary!)

$$\frac{x^7+3x+1}{x^4+x^2} = x^3 - x + \frac{x^3+3x+1}{x^4+x^2}$$

- For the remainder part, partial fractions now tells us its decomposed shape:

$$\frac{x^3+3x+1}{x^4+x^2} = \frac{x^3+3x+1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

- To find A, B, C, D, we first clear denominators:

$$x^3+3x+1 = x(x^2+1)A + (x^2+1)B + x^2(Cx+D).$$

  And then, compare the coefficients of $x^3, x^2, x, 1$ on both sides.

  [By coefficient of 1 we mean the constant terms of the polynomials (the stuff without any x).]

  For x^3: $1 = A + C$. For x^2: $0 = B + D$. For x: $3 = A$. For 1: $1 = B$.

  Hence, $A = 3$, $B = 1$, $C = 1 - A = -2$, $D = -B = -1$.

  Alternatively, we can plug in values for x to get equations in A, B, C, D. Unfortunately, these equations will be more complicated to solve (because $x = 0$ is our only super choice here...).

- Taken together we have found that: $\dfrac{x^7+3x+1}{x^4+x^2} = x^3 - x + \dfrac{3}{x} + \dfrac{1}{x^2} + \dfrac{-2x-1}{x^2+1}$.
- Finally, we can integrate to find:

$$\int \frac{x^7+3x+1}{x^4+x^2}\mathrm{d}x = \frac{1}{4}x^4 - \frac{1}{2}x^2 + 3\ln|x| - \frac{1}{x} - \ln(x^2+1) - \arctan(x) + C$$

  Here, we used that $\int \frac{1}{x^2+1}\mathrm{d}x = \arctan(x) + C$ and $\int \frac{x}{x^2+1}\mathrm{d}x = \frac{1}{2}\ln(x^2+1) + C$.

  [The last integral follows from substituting $u = x^2 + 1$. Do it!!]

Armin Straub
straub@southalabama.edu