To decompose the rational function $\frac{f(x)}{g(x)}$ into partial fractions:

(a) Check that degree $f(x) <$ degree $g(x)$. (Otherwise, long division!)

(b) Factor $g(x)$ as far as possible.

(c) For each factor of $g(x)$ collect terms as follows:

- For a linear factor $x-r$, occuring as $(x-r)^m$ in $g(x)$, these terms are

$$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \ldots + \frac{A_m}{(x-r)^m}.$$

- For a quadratic factor x^2+px+q, occuring as $(x^2+px+q)^m$ in $g(x)$, these terms are

$$\frac{B_1 x + C_1}{x^2+px+q} + \frac{B_2 x + C_2}{(x^2+px+q)^2} + \ldots + \frac{B_m x + C_m}{(x^2+px+q)^m}.$$

(d) Determine the values of the unknown constants (the A's, B's and C's).

Example 89. Evaluate $\int \frac{x^2+2}{x^3-x} \mathrm{d}x$.

Solution. The integrand is a rational function. We therefore use partial fractions.

- Note that the degree of the numerator (2) is less than the degree of the denominator (3). If this was not the case, then we would have to first do a long division.
- We need to factor the denominator: $x^3-x = x(x^2-1) = x(x-1)(x+1)$
- With two things out of the way, partial fractions now informs us that

$$\frac{x^2+2}{x^3-x} = \frac{x^2+2}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}.$$

- However, we still need to find the numbers A, B, C. To do so, we multiply the last equation by $x(x-1)(x+1)$ to get

$$x^2+2 = (x-1)(x+1)A + x(x+1)B + x(x-1)C.$$

- Setting $x=0$, we find $2=-A$. Setting $x=1$, we find $3=2B$. Setting $x=-1$, we find $3=2C$.

Alternatively, we could compare the coefficients of $x^2, x, 1$ (do it!) to get some equations in A, B, C.

- Integration now is straightforward:

$$\int \frac{x^2+2}{x^3-x} \mathrm{d}x = \int \frac{-2}{x} \mathrm{d}x + \int \frac{3/2}{x-1} \mathrm{d}x + \int \frac{3/2}{x+1} \mathrm{d}x = -2\ln|x| + \frac{3}{2}\ln|x-1| + \frac{3}{2}\ln|x+1| + C$$

Example 90. Evaluate $\int \frac{2x+1}{x^2+6x+9} \mathrm{d}x$.

- This time, after factoring, partial fractions tells us that

$$\frac{2x+1}{x^2+6x+9} = \frac{2x+1}{(x+3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2}.$$

- Clearing denominators, $2x+1 = (x+3)A + B$. Setting $x=-3$, we find $-5=B$. There is no super next choice for x so we just set $x=0$ (any other choice works as well) to get $1=3A-5$, which implies $A=2$.
- Integration now is again straightforward (make sure it is to you!):

$$\int \frac{2x+1}{x^2+6x+9} \mathrm{d}x = \int \frac{2}{x+3} \mathrm{d}x + \int \frac{-5}{(x+3)^2} \mathrm{d}x = 2\ln|x+3| + \frac{5}{x+3} + C.$$

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