Review. By integrating the product rule, we get

$$\int f(x)g'(x)\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\mathrm{d}x.$$

Example 79. $\int x\,e^x\,\mathrm{d}x =$

- To evaluate the integral, we should choose $f(x) = x$ and $g'(x) = e^x$, because $f'(x)$ becomes easier while $g(x)$ stays the same. Do it!
- If, on the other hand, we decide to choose $f(x) = e^x$ and $g'(x) = x$, then we obtain

$$\int x\,e^x\,\mathrm{d}x = \frac{1}{2}x^2\,e^x - \int \frac{1}{2}x^2 e^x\,\mathrm{d}x.$$

While certainly correct, we actually ended up with a more difficult integral.

[On the other hand, because we know $\int x\,e^x\,\mathrm{d}x$, this means we can also do $\int x^2\,e^x\,\mathrm{d}x$.]

Example 80. $\int_0^1 x^2\,e^{2x}\,\mathrm{d}x =$

After two integrations by parts, your final answer should be $\frac{1}{4}e^2 - \frac{1}{4}$.

Example 81. $\int \ln(x)\,\mathrm{d}x =$

Choose $f(x) = \ln(x)$ and $g'(x) = 1$.

Example 82. Substitute $u = \ln(x)$ in the previous integral. What do you get?

After noting that $x = e^u$, you should get $\int \ln(x)\,\mathrm{d}x = \int u\,e^u\,\mathrm{d}u$.

Then, use your answer from Example 79 and compare what you get with Example 81.

Example 83. $\int e^x \cos(x)\,\mathrm{d}x =$

We will need to integrate by parts twice.

First, let $f(x) = e^x$ and $g'(x) = \cos(x)$ (or the other way around; try it!) to get

$$\int e^x \cos(x)\,\mathrm{d}x = e^x \sin(x) - \int e^x \sin(x)\,\mathrm{d}x.$$

The new integral is of the same level of difficulty, so it might seem like we haven't gained anything. But don't give up yet! Instead, integrate by parts again with $f(x) = e^x$ and $g'(x) = \sin(x)$ to arrive at

$$\int e^x \cos(x)\,\mathrm{d}x = e^x \sin(x) - \int e^x \sin(x)\,\mathrm{d}x = e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\,\mathrm{d}x.$$

Finally, solve for $\int e^x \cos(x)\,\mathrm{d}x$!

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