A very simple model of population growth

If $y(t)$ is the size of a population (eg. of bacteria) at time t, then the rate of change $\frac{dy}{dt}$ might, from biological considerations, be (nearly) proportional to $y(t)$.

[More down to earth, this is just saying "for a population 7 times as large, we expect 7 times as many babies".]

The corresponding **mathematical model** is described by the DE $\frac{\mathrm{d}y}{\mathrm{d}t} = ky$ where k is the constant of proportionality.

[The general solution to this DE is $y(t) = Ce^{kt}$ (do it!). Hence, mathematics tells us that populations satisfying the assumption from biology necessarily exhibit exponential growth.]

Remark 75. Just to give an indication of how the modelling can be refined, let us suppose we want to take limited resources into account, so that there is a maximum sustainable population size M. This situation could be modelled by logistic equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky\left(1 - \frac{y}{M}\right).$$

Note that if y is small (compared to M), then $1 - \frac{y}{M} \approx 1$, so $\frac{\mathrm{d}y}{\mathrm{d}t} \approx ky$, and we are back at our previous model. However, once the population is getting close to M then $1 - \frac{y}{M} \approx 0$, so $\frac{\mathrm{d}y}{\mathrm{d}t} \approx 0$, which means that the population does not continue to grow.

[Main problem of modeling: a model has to be detailed enough to resemble the real world, yet simple enough to allow for mathematical analysis.]

Example 76. A yeast culture, with initial mass 12 g, is assumed to exhibit exponential growth. After 10 min, the mass is 15 g. What is the mass after t min?

Solution. Let $y(t)$ be the mass after t min. Then, $y(t) = 12e^{kt}$ with $k = \frac{1}{10}\ln\left(\frac{5}{4}\right)$.

[Do you see that this can be simplified to $y(t) = 12\,(5/4)^{t/10}$?]

Integration by parts

Integration by parts is the reversal of the product rule:

$$\begin{aligned} (fg)' &= f'g + fg' \\ \overset{\text{antiderivative}}{\rightsquigarrow} \quad f(x)g(x) &= \int f'(x)g(x)\mathrm{d}x + \int f(x)g'(x)\mathrm{d}x \end{aligned}$$

$$\boxed{\int f(x)g'(x)\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\mathrm{d}x}$$

The following shorthand is very common: (here $u = f(x)$, $v = g(x)$ so that $\mathrm{d}u = f'(x)\mathrm{d}x$ and $\mathrm{d}v = g'(x)\mathrm{d}x$)

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$$

Example 77. $\int x\cos(x)\,\mathrm{d}x = x\sin(x) - \int 1 \cdot \sin(x)\,\mathrm{d}x = x\sin(x) + \cos(x) + C$

Here, we chose $f(x) = x$ and $g'(x) = \cos(x)$, so that $g(x) = \sin(x)$ (we are free to choose the simplest antiderivative).

Example 78. $\int x\,e^x\,\mathrm{d}x =$

Armin Straub
straub@southalabama.edu