Differential equations

Example 62. The **differential equation** $\frac{dy}{dx} = y$ is solved by $y(x) = e^x$. It is also solved by $y(x) = 0$ and $y(x) = 7e^x$. Its general solution is $y(x) = Ce^x$ where C can be any number.

Example 63. The **initial value problem** $\frac{dy}{dx} = y$, $y(0) = 1$ has the unique solution $y(x) = e^x$.

The fact that the exponential function solves these simple equations is at the root of why it is so important!

Example 64. The general solution to the differential equation (DE) $\frac{dy}{dx} = x^2$ is $y(x) = \frac{1}{3}x^3 + C$.

[So, computing the antiderivative of $f(x)$ is the same as solving the (very special) DE $\frac{dy}{dx} = f(x)$.]

Example 65. Verify that the differential equation $\frac{dy}{dx} = y^2$ is solved by

- $y(x) = -\frac{1}{x}$,
- $y(x) = -\frac{1}{x+3}$,
- $y(x) = 0$.

Its general solution is $y(x) = -\frac{1}{x+C}$. (The solution $y = 0$ corresponds to $C \to \infty$.)

Example 66. Solve the IVP $\frac{dy}{dx} = y^2$, $y(0) = 2$.

[Using, for now, the general solution from the previous example.]

The next example demonstrates the method of **separation of variables** to solve (a certain class of) differential equations.

Example 67. Let us solve the DE $\frac{dy}{dx} = y^2$ by separation of variables.

[Some of the next steps might feel questionable... However, as illustrated above, we can always verify afterwards that we indeed found a solution.]

In the first step, we separate the variables:

$$\frac{1}{y^2}\, dy = dx$$

[If the DE is of the form $\frac{dy}{dx} = g(x)h(y)$, then we would separate it as $\frac{1}{h(y)}\, dy = g(x)\, dx$.]

We then integrate both sides and compute the indefinite integrals:

$$\begin{aligned} \int \frac{1}{y^2}\, dy &= \int dx \\ -\frac{1}{y} &= x + C \end{aligned}$$

[we combine the two constants of integration into one]

If possible (like here) we solve the resulting equation for y:

$$y = -\frac{1}{x+C}$$