Example 44. What is the length of the curve $y=2x$, for $0\leqslant x\leqslant 4$?

Make a sketch and use Pythagoras.

The length of a general curve $y=f(x)$, for $a\leqslant x\leqslant b$, is given by

$$\int_a^b \sqrt{1+(f'(x))^2}\,\mathrm{d}x.$$

To see how we can arrive at this formula, we proceeded as follows:

- We will chop the x-axis into little pieces of width $\mathrm{d}x$ and look at the corresponding pieces of our graph.
- Suppose we are looking at our graph near x. If we zoom in plenty, then the tiny portion of the graph we see begins to look roughly like a line with slope $f'(x)$.
- We can compute the length of a segment of this line as we did in Example 44 by using Pythagoras. If the segment extends $\mathrm{d}x$ horizontally, then it extends $f'(x)\mathrm{d}x$ vertically (make a sketch!). [We can also write $f'(x)\mathrm{d}x=\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x=\mathrm{d}y$.]

  By Pythagoras, our piece of the line has length

$$\sqrt{(\mathrm{d}x)^2+(f'(x)\,\mathrm{d}x)^2}=\sqrt{1+(f'(x))^2}\,\mathrm{d}x.$$

- "Adding" all these little "lines", we arrive at the formula above for the total length of the curve.

Example 45. Using the integral formula, compute the length of the curve $y=2x$, for $0\leqslant x\leqslant 4$, again. Of course, the answer agrees with Example 44.

Example 46. Compute the length of the curve $y=x^{3/2}$, for $0\leqslant x\leqslant 4$.

Make a sketch! Which curve should be longer, $y=x^{3/2}$ or $y=2x$? Compare the lengths numerically (should be 8.944 and 9.073).

One step of our computations in class was

$$\int_0^4 \sqrt{1+\frac{9}{4}x}\,\mathrm{d}x=\int_1^{10}\frac{4}{9}\sqrt{u}\,\mathrm{d}u,$$

where we substituted $u=\frac{9}{4}x$. Make sure that this substitution is crystal-clear to you (including the change of boundaries: if $x=4$ then $u=?$).

Example 47. Setup an integral for the circumference of a circle of radius r.

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