Homework #9

MATH 126 — Calculus II
Friday, Sept 18

Please print your name:

Problem 1. (9.2.24) Express the number $1.\overline{414} = 1.414414414...$ as the ratio of two integers.

Solution. $1.\overline{414} = 1 + \frac{414}{1000} + \frac{414}{1000^2} + ... = 1 + \frac{414}{1000}\sum_{n=0}^{\infty} 1000^{-n} = 1 + \frac{414}{1000}\frac{1}{1 - \frac{1}{1000}} = 1 + \frac{414}{999} = \frac{1413}{999} = \frac{157}{111}$ □

Problem 2. (9.2.56) Does the series $\sum_{n=1}^{\infty} \log\frac{1}{3^n}$ converge?

Solution. Note that $\log\frac{1}{3^n} = -n\log(3) \to -\infty \neq 0$ as $n \to \infty$. Therefore, the series $\sum_{n=1}^{\infty} \log\frac{1}{3^n}$ diverges. □

Problem 3. (9.3.6) Use the integral test to determine if the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$ converges or diverges.

Solution. By the integral test, the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$ converges if and only if the integral $\int_2^{\infty} \frac{\mathrm{d}x}{x(\log x)^2}$ converges.

First, however, we should verify that the integral test indeed applies: the function $\frac{1}{x(\log x)^2}$ is obviously positive and continuous for $x \geqslant 2$. It is also decreasing, because $x(\log x)^2$ clearly increases.

Upon substituting $u = \log x$, we find that

$$\int_2^{\infty} \frac{\mathrm{d}x}{x(\log x)^2} = \int_{\log(2)}^{\infty} \frac{\mathrm{d}u}{u^2} = \left[-\frac{1}{u}\right]_{\log(2)}^{\infty}$$

is finite because $\lim_{u \to \infty}\left(-\frac{1}{u}\right) = 0$. Therefore, the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$ converges. □

Problem 4. (9.3.38) Does the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ converge?

Solution. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ converges if and only if $\int_1^{\infty} \frac{x}{x^2+1}\mathrm{d}x$ converges.

First, however, we should verify that the integral test indeed applies: the function $\frac{x}{x^2+1}$ is obviously positive and continuous for $x \geqslant 1$. It is also decreasing, because $\frac{x^2+1}{x} = x + \frac{1}{x}$ is increasing (its derivative is $1 - \frac{1}{x^2}$, which is positive if $x > 1$).

Substituting $u = x^2 + 1$, we find

$$\int_1^{\infty} \frac{x}{x^2+1}\mathrm{d}x = \frac{1}{2}\int_2^{\infty} \frac{\mathrm{d}u}{u} = [\ln|u|]_2^{\infty} = \infty$$

because $\lim_{u \to \infty} \ln|u| = \infty$. Hence $\int_1^{\infty} \frac{x}{x^2+1}\mathrm{d}x$ diverges, and we conclude that $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges.

Comment. Note that it is more convenient to show the divergence of this series using the limit comparison test. Do it! □

Problem 5. (9.4.6) Use the comparison test to determine if the series $\sum_{n=1}^{\infty} \frac{1}{n3^n}$ converges.

Solution. Note that $n3^n \geqslant 3^n$ for all $n \geqslant 1$. Hence, $\sum_{n=1}^{\infty} \frac{1}{n3^n} \leqslant \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{3}\frac{1}{1 - \frac{1}{3}} = \frac{1}{2} < \infty$. In particular, our series converges. □

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