Homework #8

MATH 126 — Calculus II
Friday, Oct 23

Please print your name:

Problem 1. (9.1.36, 9.1.46, 9.1.62) Which of the following sequences $\{a_n\}$ converge, and which diverge? Find the limit of each convergent sequence.

(a) $a_n = (-1)^n\left(1 - \frac{1}{n}\right)$

(b) $a_n = \frac{\sin^2 n}{2^n}$

(c) $a_n = \sqrt[n]{3^{2n+1}}$

Solution.

(a) $\lim_{n\to\infty} a_n$ does not exist.
That's because $\lim_{n\to\infty}\left(1 - \frac{1}{n}\right) = 1$ while $\lim_{n\to\infty}(-1)^n$ does not exist.

(b) Note that $0 \leqslant a_n \leqslant \frac{1}{2^n}$. Since $\lim_{n\to\infty}\frac{1}{2^n} = 0$, it follows that $\lim_{n\to\infty} a_n = 0$ (because a_n is squeezed between two sequences that both converges to 0).

(c) Note that $a_n = \sqrt[n]{3^{2n+1}} = 3^{\frac{2n+1}{n}} = 3^{2+\frac{1}{n}}$. Since $2 + \frac{1}{n} \to 2$ as $n \to \infty$ (and the function 3^x is continuous) it follows that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 3^{2+\frac{1}{n}} = 3^2 = 9$. □

Problem 2. (9.1.98) Assume that the sequence

$$\sqrt{1},\quad \sqrt{1+\sqrt{1}},\quad \sqrt{1+\sqrt{1+\sqrt{1}}},\quad \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}},\quad \ldots$$

converges and find its limit.

Solution. This sequence $\{a_n\}$ is defined recursively: $a_1 = \sqrt{1}$ and $a_n = \sqrt{1 + a_{n-1}}$ for $n \geqslant 2$.

- Suppose that $\lim_{n\to\infty} a_n = L$. Taking the limit of both sides of $a_n = \sqrt{1 + a_{n-1}}$, we get

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty}\sqrt{1 + a_{n-1}} = \sqrt{1+L}.$$

- Writing $L = \sqrt{1+L}$ as $L^2 = 1 + L$ and solving this quadratic equation shows that $L = \frac{1 \pm \sqrt{5}}{2}$.

- Since $\frac{1-\sqrt{5}}{2}$ is negative (and our sequence is positive), the limit (if it exists) has to be $L = \frac{1+\sqrt{5}}{2}$. □

Problem 3. (Example 129) Assume that the sequence

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \ldots$$

converges and find its limit.

Solution. Recall that 1, 1, 2, 3, 5, 8, 13, 21, ... are the *Fibonacci numbers* $\{F_n\}$. They are defined *recursively*: $F_1 = 1$, $F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 3$.

- Our sequence are quotients of Fibonacci numbers $\{a_n\}$ with $a_n = \frac{F_{n+1}}{F_n}$.
- Take $F_{n+1} = F_n + F_{n-1}$ and divide both sides by F_n to get the recursive relation $a_n = 1 + \frac{1}{a_{n-1}}$.
- Suppose our sequence converges and $\lim_{n\to\infty} a_n = L$. Taking the limit of both sides of $a_n = 1 + \frac{1}{a_{n-1}}$, we get
$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1 + \frac{1}{a_{n-1}}\right) = 1 + \frac{1}{L}.$$
- Writing $L = 1 + \frac{1}{L}$ as $L^2 = L + 1$ and solving this quadratic equation shows that $L = \frac{1 \pm \sqrt{5}}{2}$.
- Since $\frac{1-\sqrt{5}}{2}$ is negative (and our sequence is positive), the limit (if it exists) has to be $L = \frac{1+\sqrt{5}}{2}$.
- Numerically, $L = \frac{1+\sqrt{5}}{2} \approx 1.618$. The first few terms of our sequence are 1, 2, 1.5, 1.667, 1.6, 1.625, 1.615, 1.619, ..., which convinces us that our limit is correct. □

Problem 4. (9.2.8, 9.2.12) Write out the first eight terms of each series to show how the series starts. Then find the sum of the series or show that it diverges.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{4^n}$

(b) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{5}{2^n} - \frac{1}{3^n}\right)$

Solution.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{4^n} = \sum_{n=0}^{\infty} \frac{1}{4^n} - \left(1 + \frac{1}{4}\right) = \frac{1}{1-\frac{1}{4}} - \frac{5}{4} = \frac{1}{12}$

or: $\displaystyle\sum_{n=2}^{\infty} \frac{1}{4^n} = \frac{1}{4^2}\sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{1}{16}\,\frac{1}{1-\frac{1}{4}} = \frac{1}{12}$

[The first 8 terms are: $\frac{1}{16}, \frac{1}{64}, \frac{1}{256}, \frac{1}{1024}, \frac{1}{4096}, \frac{1}{16384}, \frac{1}{65536}, \frac{1}{262144}$]

(b) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{5}{2^n} - \frac{1}{3^n}\right) = 5\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n - \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n = 5\frac{1}{1-\frac{1}{2}} - \frac{1}{1-\frac{1}{3}} = 10 - \frac{3}{2} = \frac{17}{2}$

[The first 8 terms are: $4, \frac{13}{6}, \frac{41}{36}, \frac{127}{216}, \frac{389}{1296}, \frac{1183}{7776}, \frac{3581}{46656}, \frac{10807}{279936}$] □

Problem 5. (9.2.76) Find the values of x for which the geometric series

$$\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n (x-3)^n$$

converges. Also, find the sum of the series (as a function of x) for those values of x.

Solution. $\displaystyle\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n (x-3)^n = \sum_{n=0}^{\infty}\left(\frac{3-x}{2}\right)^n = \frac{1}{1-\frac{3-x}{2}} = \frac{2}{x-1}$

provided that $\left|\frac{3-x}{2}\right| < 1$ or, equivalently, $1 < x < 5$. □

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