Homework #7

MATH 126 — Calculus II
Friday, Oct 16

Please print your name:

Problem 1. (Example 114) Evaluate the indefinite integral $\int \frac{1}{(1+x^2)^2}\mathrm{d}x$.

Solution. We substitute $x = \tan\theta$ because then $1 + x^2 = \sec^2\theta$.

Recall that $\frac{\mathrm{d}x}{\mathrm{d}\theta} = \sec^2\theta$. Hence, $\mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$ and we find

$$\int \frac{\mathrm{d}x}{(1+x^2)^2} = \int \frac{\sec^2\theta\,\mathrm{d}\theta}{\sec^4\theta} = \int \cos^2\theta\,\mathrm{d}\theta.$$

By the usual integration by parts, followed by using $\cos^2\theta + \sin^2\theta = 1$,

$$\int \cos^2\theta\,\mathrm{d}x = \sin\theta\cos\theta + \int \sin^2\theta\,\mathrm{d}\theta = \sin\theta\cos\theta + \theta - \int \cos^2\theta\,\mathrm{d}\theta.$$

We conclude that $\int \cos^2\theta\,\mathrm{d}\theta = \frac{\theta + \sin\theta\cos\theta}{2} + C$. Hence,

$$\int \frac{\mathrm{d}x}{(1+x^2)^2} = \int \cos^2\theta\,\mathrm{d}\theta = \frac{\theta + \sin\theta\cos\theta}{2} + C = \frac{1}{2}[\arctan(x) + \sin(\arctan(x))\cos(\arctan(x))] + C.$$

That's an acceptable answer. However, it turns out that it can be simplified to

$$\int \frac{\mathrm{d}x}{(1+x^2)^2} = \frac{1}{2}\left[\arctan(x) + \frac{x}{1+x^2}\right] + C.$$

To realize that, we need to see that

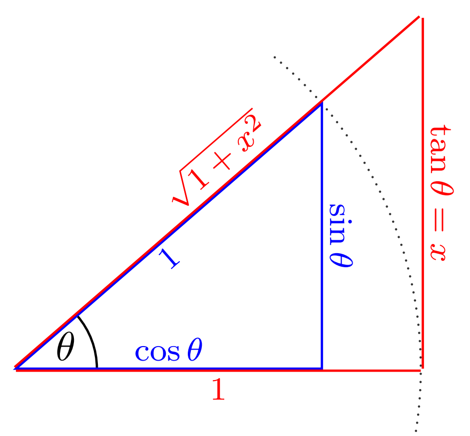
$$\sin(\arctan(x)) = \sin\theta = \frac{x}{\sqrt{1+x^2}}$$

and

$$\cos(\arctan(x)) = \cos\theta = \frac{1}{\sqrt{1+x^2}}.$$

This can be best observed from the picture to the right: note the similar triangles in the picture and conclude that

$$\frac{\sin\theta}{1} = \frac{x}{\sqrt{1+x^2}} \quad \text{and} \quad \frac{\cos\theta}{1} = \frac{1}{\sqrt{1+x^2}}.$$



[By the way, if you decided to use the trig identity $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$ to find

$$\int \cos^2\theta\,\mathrm{d}\theta = \int \frac{1+\cos(2\theta)}{2}\mathrm{d}\theta = \frac{\theta}{2} + \frac{1}{4}\sin(2\theta),$$

then, when substituting back, you would need the trig identity $\sin(2\theta) = 2\sin\theta\cos\theta$ to get the same simplified answer.] □

Armin Straub
straub@southalabama.edu

Problem 2. (4.5.12, 4.5.46) Determine the following limits:

(a) $\lim_{x\to\infty} \frac{x-8x^2}{12x^2+5x}$

(b) $\lim_{x\to\infty} x^2 e^{-x}$

Solution.

(a) We cancel an x to get $\lim_{x\to\infty} \frac{x-8x^2}{12x^2+5x} = \lim_{x\to\infty} \frac{1-8x}{12x+5}$.

The limit resulting limit is still in the indeterminate form "$\frac{\infty}{\infty}$". We may therefore apply L'Hospital to find

$$\lim_{x\to\infty} \frac{x-8x^2}{12x^2+5x} = \lim_{x\to\infty} \frac{1-8x}{12x+5} \underset{\text{L'H.}}{\overset{\text{“}\frac{\infty}{\infty}\text{”}}{=\!=}} \lim_{x\to\infty} \frac{-8}{12} = -\frac{8}{12} = -\frac{2}{3}.$$

(b) We rewrite the limit as $\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x}$ so that it is in the indeterminate form "$\frac{\infty}{\infty}$". We may then apply L'Hospital to find

$$\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x} \underset{\text{L'H.}}{\overset{\text{“}\frac{\infty}{\infty}\text{”}}{=\!=}} \lim_{x\to\infty} \frac{2x}{e^x} \underset{\text{L'H.}}{\overset{\text{“}\frac{\infty}{\infty}\text{”}}{=\!=}} \lim_{x\to\infty} \frac{2}{e^x} = 0.$$

□

Problem 3. (9.1.32, 9.1.34, 9.1.55) Which of the following sequences $\{a_n\}$ converge, and which diverge? Find the limit of each convergent sequence.

(a) $a_n = \frac{n+3}{n^2+5n+6}$

(b) $a_n = \frac{1-n^3}{70-4n^2}$

(c) $a_n = \sqrt[n]{10n}$

Solution.

(a) The sequence converges and $\lim_{n\to\infty} a_n = 0$.

This follows from the usual application of L'Hospital or by rewriting the sequence as

$$a_n = \frac{n+3}{n^2+5n+6} = \frac{\frac{1}{n} + \frac{3}{n^2}}{1 + \frac{5}{n} + \frac{6}{n^2}}.$$

(b) The sequence diverges and $\lim_{n\to\infty} a_n = \infty$. This follows as in the first part.

(c) We first look at the sequence $\log(a_n) = \frac{\log(10n)}{n}$. Since this is in the indeterminate form "$\frac{\infty}{\infty}$", we may apply L'Hospital to find

$$\lim_{n\to\infty} \log(a_n) = \lim_{n\to\infty} \frac{\log(10n)}{n} == \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0.$$

Finally, we conclude that the original sequence converges and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\log(a_n)} = e^0 = 1.$$

□