Homework #6

MATH 126 — Calculus II
Friday, Oct 2

Please print your name:

Problem 1. (8.4.20) Evaluate $\int \frac{x^2\,\mathrm{d}x}{(x-1)(x^2+2x+1)}$.

Solution. Note that the degree of the numerator is less than the degree of the denominator. Partial fractions therefore predicts that

$$\frac{x^2}{(x-1)(x^2+2x+1)} = \frac{x^2}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

To find A, B, C, we clear denominators to get

$$x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1).$$

Setting $x=1$ gives $1=4A$, and setting $x=-1$ gives $1=-2C$. Hence, $A=\frac{1}{4}$ and $C=-\frac{1}{2}$. There is no third super choice for x, so we just plugin some value, say, $x=0$. That gives $0=A-B-C$, which we solve for B to get $B=A-C=\frac{1}{4}-\left(-\frac{1}{2}\right)=\frac{3}{4}$.

We can now integrate:

$$\int \frac{x^2\,\mathrm{d}x}{(x-1)(x^2+2x+1)} = \int \left[\frac{1/4}{x-1} + \frac{3/4}{x+1} + \frac{-1/2}{(x+1)^2}\right]\mathrm{d}x = \frac{1}{4}\log(x-1) + \frac{3}{4}\log(x+1) + \frac{1}{2}\frac{1}{x+1} + C.$$

□

Problem 2. (8.4.53) Solve the initial value problem

$$(t^2+2t)\frac{\mathrm{d}x}{\mathrm{d}t} = 2x+2 \quad (t, x > 0), \quad x(1) = 1.$$

Solution. We separate variables to get

$$\frac{1}{2x+2}\,\mathrm{d}x = \frac{1}{t^2+2t}\,\mathrm{d}t \quad \Longrightarrow \quad \int \frac{1}{2x+2}\,\mathrm{d}x = \int \frac{1}{t^2+2t}\,\mathrm{d}t.$$

The second integral can be computed by partial fractions:

$$\frac{1}{t^2+2t} = \frac{1}{t(t+2)} = \frac{A}{t} + \frac{B}{t+2}.$$

To find A, B, we clear denominators to get $1=A(t+2)+Bt$. Setting $t=0$ gives $1=2A$, and setting $t=-2$ gives $1=-2B$. Hence, $A=\frac{1}{2}$ and $B=-\frac{1}{2}$.

We continue with $\int \frac{1}{2x+2}\,\mathrm{d}x = \int \frac{1}{t^2+2t}\,\mathrm{d}t$, which becomes:

$$\frac{1}{2}\log(x+1) = \int \left[\frac{1/2}{t} + \frac{-1/2}{t+2}\right]\mathrm{d}t = \frac{1}{2}[\log(t) - \log(t+2)] + C = \frac{1}{2}\log\frac{t}{t+2} + C. \tag{1}$$

Setting $x=1$ and $t=1$, we obtain $\frac{1}{2}\log(2) = \frac{1}{2}\log\left(\frac{1}{3}\right) + C$, which we solve to get $C = \frac{1}{2}\left[\log(2) - \log\left(\frac{1}{3}\right)\right] = \frac{1}{2}\log(6)$.

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Solving (1) for x, we finally find

$$\log(x+1) = \log\frac{t}{t+2} + \log 6 = \log\frac{6t}{t+2} \quad \Longrightarrow \quad x = \frac{6t}{t+2} - 1 = \frac{5t-2}{t+2}.$$

□

Problem 3. (8.7.24) Evaluate $\int_{-\infty}^{\infty} 2x e^{-x^2} \mathrm{d}x$.

Solution. Let us first compute the indefinite integral $\int 2x e^{-x^2} \mathrm{d}x$ by substituting $u = x^2$ (so that $\mathrm{d}u = 2x\mathrm{d}x$).

$$\int 2x e^{-x^2} \mathrm{d}x = \int e^{-u} \mathrm{d}u = -e^{-u} + C = -e^{-x^2} + C$$

Hence,

$$\int_{-\infty}^{\infty} 2x e^{-x^2} \mathrm{d}x = \left[-e^{-x^2}\right]_{-\infty}^{\infty} = 0 - 0 = 0.$$

[Note that we could have anticipated to get zero from the fact that the integrand is an odd function.] □

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