Homework #4

MATH 126 — Calculus II
Friday, Sept 18

Please print your name:

Problem 1. (6.5.32a, Forcing electrons together) Two electrons r meters apart repel each other with a force of

$$F = \frac{23 \cdot 10^{-29}}{r^2} \text{ newtons.}$$

Suppose one electron is held fixed at the point $(1, 0)$ on the x-axis (units in meters). How much work does it take to move a second electron along the x-axis from the point $(-1, 0)$ to the origin?

Solution. Consider the moment when the second electron is at position $(x, 0)$. Because its distance to the first electron is $r = 1 - x$, the amount of work it takes to move it $\mathrm{d}x$ units towards the origin is (roughly)

$$Fd = \frac{23 \cdot 10^{-29}}{(1-x)^2} \mathrm{d}x.$$

"Adding" up these little amounts of work from $x = -1$ to $x = 0$, we find that the total amount of work is

$$\int_{-1}^{0} \frac{23 \cdot 10^{-29}}{(1-x)^2} \mathrm{d}x = 23 \cdot 10^{-29} \left[\frac{1}{1-x} \right]_{-1}^{0} = 23 \cdot 10^{-29} \cdot \left(1 - \frac{1}{2} \right) = 11.5 \cdot 10^{-29} \text{ Nm.}$$

□

Problem 2. (6.5.16, Pumping a half-full tank) Consider the conical tank which is the solid of revolution resulting from revolving the area enclosed by $y = 2x$, $x = 0$, $y = 10$ (units in feet) about the y-axis (see Figure 6.39 in book). Suppose that this tank is only filled to half its height with olive oil weighing 57 lb/ft^3. How much work does it take to pump the remaining oil to a level 4 ft above the top of the tank?

Solution. Consider a horizontal slice of the oil at height y and thickness $\mathrm{d}y$. As in Example 5, this slice has volume

$$\pi \left(\frac{1}{2} y \right)^2 \mathrm{d}y = \frac{\pi}{4} y^2 \, \mathrm{d}y \quad \text{ft}^3$$

and therefore weighs $\frac{57\pi}{4} y^2 \, \mathrm{d}y$ lb. This slice needs to be lifted $10 + 4 - y$ ft, which takes

$$(14 - y) \frac{57\pi}{4} y^2 \, \mathrm{d}y \quad \text{ft-lb}$$

of work. There are slices from $y = 0$ to $y = 5$ (half the total height of 10), so that the total amount of work is

$$\int_0^5 (14 - y) \frac{57\pi}{4} y^2 \, \mathrm{d}y = \frac{57\pi}{4} \int_0^5 (14 - y) y^2 \, \mathrm{d}y = \frac{57\pi}{4} \int_0^5 (14y^2 - y^3) \, \mathrm{d}y = \frac{57\pi}{4} \left[\frac{14}{3} y^3 - \frac{1}{4} y^4 \right]_0^5 = \frac{57\pi}{4} \cdot \frac{5125}{12} \approx 19120 \text{ ft-lb.}$$

[The problem as stated in the book may also be interpreted so that the tank is filled to half its volume. The volume up to height y is $\frac{1}{3}\pi\left(\frac{1}{2}y\right)^2 y = \frac{\pi}{12} y^3$, so that the total volume (for $y = 10$) is $V = \frac{1000\pi}{12}$. To find at which height the tank is half full, we need to solve $\frac{\pi}{12} y^3 = \frac{V}{2} = \frac{500\pi}{12}$, which simplifies to $y^3 = 500$, and so $y = \sqrt[3]{500}$. With the same reasoning as before, the total work is now

$$\int_0^{\sqrt[3]{500}} (14 - y) \frac{57\pi}{4} y^2 \, \mathrm{d}y = \frac{57\pi}{4} \left[\frac{14}{3} y^3 - \frac{1}{4} y^4 \right]_0^{\sqrt[3]{500}} \approx 60043 \text{ ft-lb.]}$$

□

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Problem 3. (7.2.10) Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2\sqrt{y}, \quad y > 0.$$

Solution. Separating variables, we obtain

$$\frac{1}{\sqrt{y}}\,\mathrm{d}y = x^2\,\mathrm{d}x.$$

We then integrate to find

$$\int \frac{1}{\sqrt{y}}\,\mathrm{d}y = \int x^2\,\mathrm{d}x \quad \Longrightarrow \quad 2\sqrt{y} = \frac{1}{3}x^3 + C \quad \Longrightarrow \quad y = \left(\frac{1}{6}x^3 + \frac{C}{2}\right)^2.$$

[If we feel like it, we can replace $C/2$ by some new constant D.] □

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