Homework #3

MATH 126 — Calculus II
Friday, Sept 11

Please print your name:

Problem 1. (6.3.26) The graph of the equation $x^{2/3} + y^{2/3} = 1$ is one of a family of curves called *astroids* (not "asteroids") because of their starlike appearance (see the accompanying figure in the book). Find the length of this particular astroid by finding the length of half the first-quadrant portion,

$$y = (1 - x^{2/3})^{3/2}, \quad \sqrt{2}/4 \leqslant x \leqslant 1,$$

and multiplying by 8.

Solution. The length of 1/8th of the astroid is

$$\begin{aligned} \int_{\sqrt{2}/4}^{1} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x &= \int_{\sqrt{2}/4}^{1} \sqrt{1 + \left(\frac{3}{2}(1 - x^{2/3})^{1/2}\left(-\frac{2}{3}x^{-1/3}\right)\right)^2}\,\mathrm{d}x \\ &= \int_{\sqrt{2}/4}^{1} \sqrt{1 + (1 - x^{2/3})x^{-2/3}}\,\mathrm{d}x = \int_{\sqrt{2}/4}^{1} x^{-1/3}\,\mathrm{d}x \\ &= \left[\frac{3}{2}x^{2/3}\right]_{\sqrt{2}/4}^{1} = \frac{3}{2}\left(1 - \left(\frac{1}{8}\right)^{1/3}\right) = \frac{3}{4}. \end{aligned}$$

Hence, the length of the entire astroid is 6. □

Problem 2. (6.5.8) A bag of sand originally weighing 144 lb was lifted at a constant rate. As it rose, sand also leaked out a constant rate. The sand was half gone by the time the bag had been lifted to 18 ft. How much work was done lifting the sand this far? (Neglect the weight of the bag and lifting equipment.)

Solution. We have to lift a total of 18 ft. Let us consider the moment when the bag has already been lifted x feet and try to figure out the amount of work to lift it an additional $\mathrm{d}x$ feet.

- Since sand is leaking at a constant rate, the bag only contains $\left(144 - 72 \cdot \frac{x}{18}\right) = (144 - 4x)$ lb of sand at our moment.

  [If you have trouble with this step, note that sand is leaking out at the rate of $\left(\frac{1}{2} \cdot 144 \text{ lb}\right)/(18 \text{ ft})$.]

- Hence, it takes $(144 - 4x) \cdot \mathrm{d}x$ ft-lb to lift the bag an additional $\mathrm{d}x$ feet.

The total amount of work is

$$\int_0^{18} (144 - 4x) \cdot \mathrm{d}x = [144x - 2x^2]_0^{18} = 1944 \text{ ft-lb}.$$

□

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