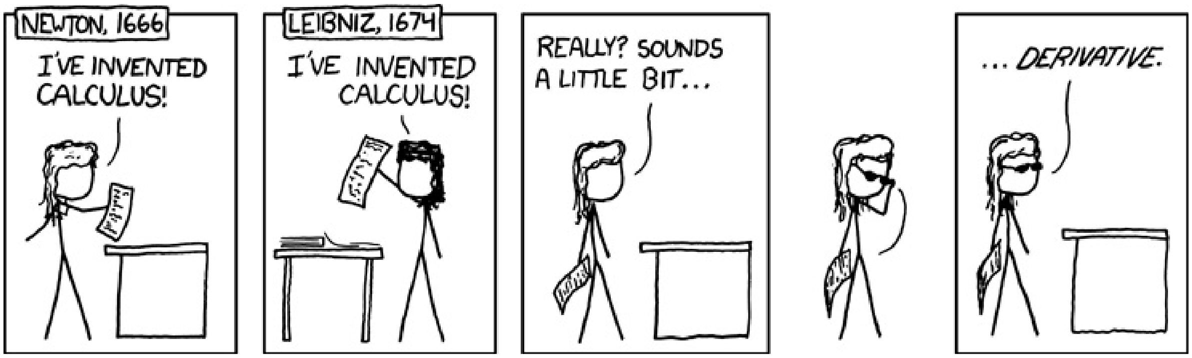
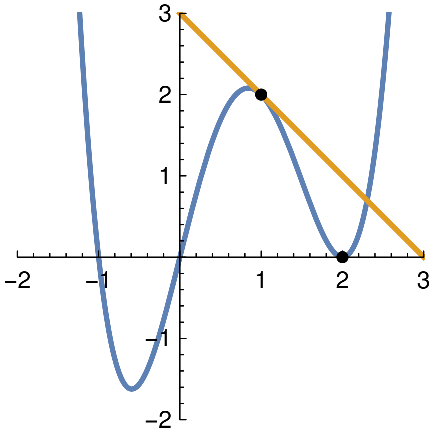


https://xkcd.com/626/

With $f(x)$ as in the graph, estimate:



$f(1) = 2$

$f'(1) = -1$

$f(2) = 0$

$f'(2) = 0$

If $f(x) = x^4 - 3x^3 + 4x$, (that's the function in the plot)

then $f'(x) = 4x^3 - 9x^2 + 4$.

In particular, $f'(1) = 4 - 9 + 4 = -1$ and $f'(2) = 4 \cdot 8 - 9 \cdot 4 + 4 = 0$.

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1 Extrema

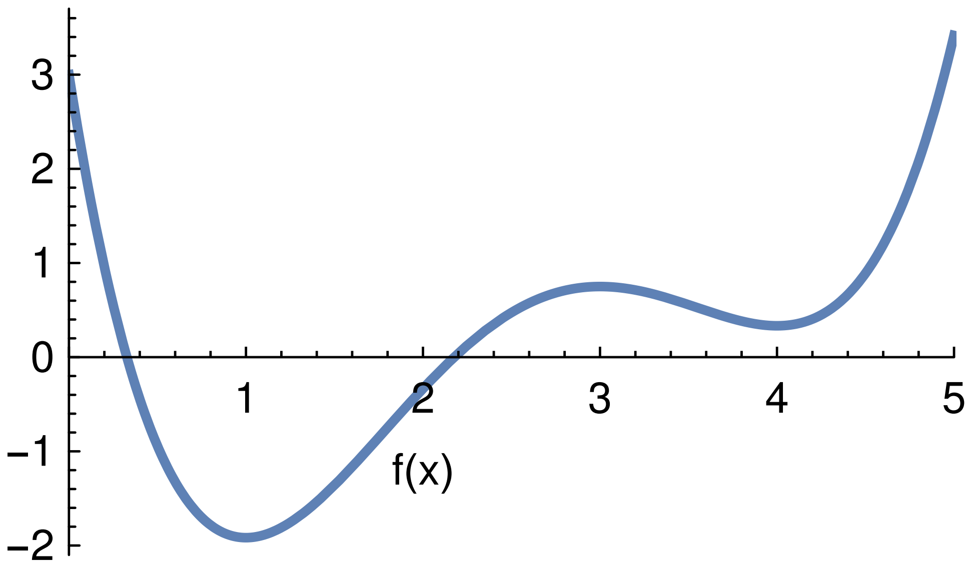
We will be very interested in extrema (maxima or minima):

- **(absolute) extrema**

  Here, the function is higher/lower than at any other point.

- **local extrema** (also called relative extrema)

  Here, the function is higher/lower than at nearby points.



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Example 1. List all extrema of the above function $f(x)$.

(a) Local minima: at $x=1$ and at $x=4$

(b) Local maxima: at $x=3$

(c) Absolute minimum: at $x=1$

(d) Absolute maximum:

It does not look like $f(x)$ has an absolute maximum (instead the values for $x>5$ or $x<0$ seem to be growing without bound).

On the other hand, if the domain of $f(x)$ is only the interval $[0,5]$ (that is, $f(x)$ is only defined for $0 \leqslant x \leqslant 5$), then the absolute maximum is at $x=5$.

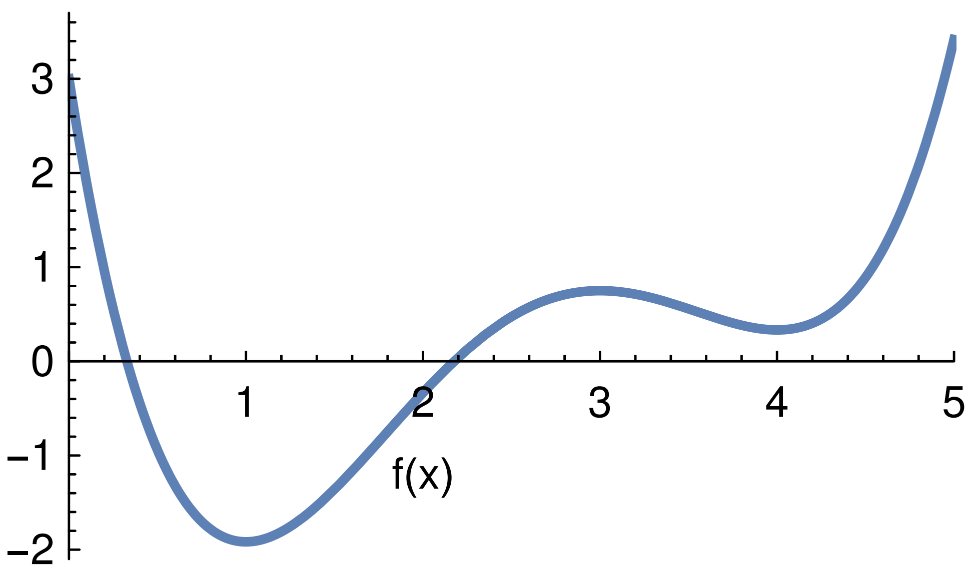
(first-derivative rule)

- If $f'(a)>0$, then $f(x)$ is increasing at $x=a$.
- If $f'(a)<0$, then $f(x)$ is decreasing at $x=a$.
- If $f'(a)=0$, then $f(x)$ might have a relative extremum at $x=a$.

Such a (where $f'(a)=0$) are called **critical values**.

However, we need to investigate further these are indeed extrema.

(That's the point of the first- and second-derivative tests.)



Example 2. On which intervals is $f(x)$ increasing/decreasing?

(a) $f(x)$ is increasing for: $1<x<3$ and $x>4$

(b) $f(x)$ is decreasing for: $x<1$ and $3<x<4$

(c) Sketch $f'(x)$.

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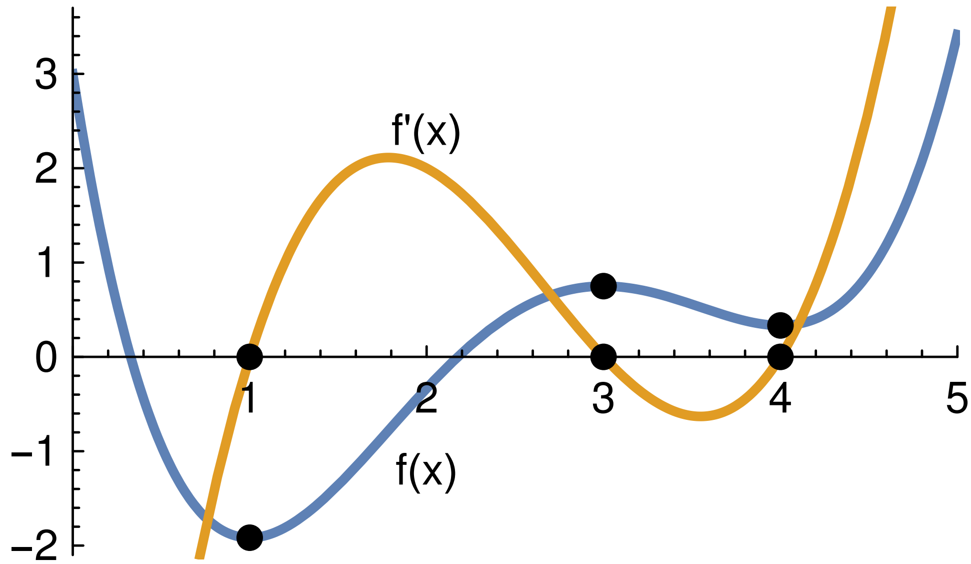


2 Concavity

Example 3. For the same $f(x)$, describe the slopes between $x=1$ and $x=3$.

For $1<x<3$, the slopes are positive (i.e. $f(x)$ is increasing).

But we can say more:

The slopes are increasing from $x=1$ until $x\approx 1.8$ (the maximal slope is about 2.1), then the slopes are descreasing from $x\approx 1.8$ to $x=3$.

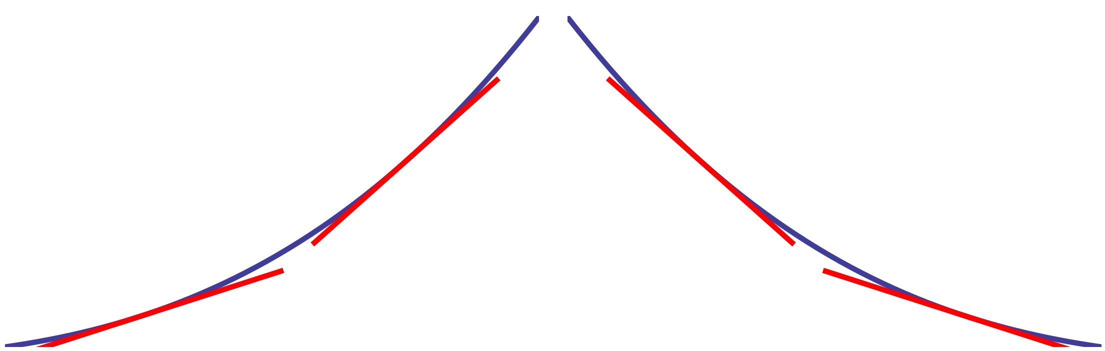
The point $x\approx 1.8$ is special. It is an **inflection point** (see below).

In any case, slopes are changing. It is of interest whether slopes are increasing or decreasing.

$f(x)$ is **concave up** (at $x=a$):

$\Longleftrightarrow$ graph lies above tangent line (around $x=a$)

$\Longleftrightarrow$ $f'(x)$ is increasing (at $x=a$) (i.e. slopes are increasing)



Being concave down is defined analogously.

Recall that derivatives can tell us whether a function is increasing!

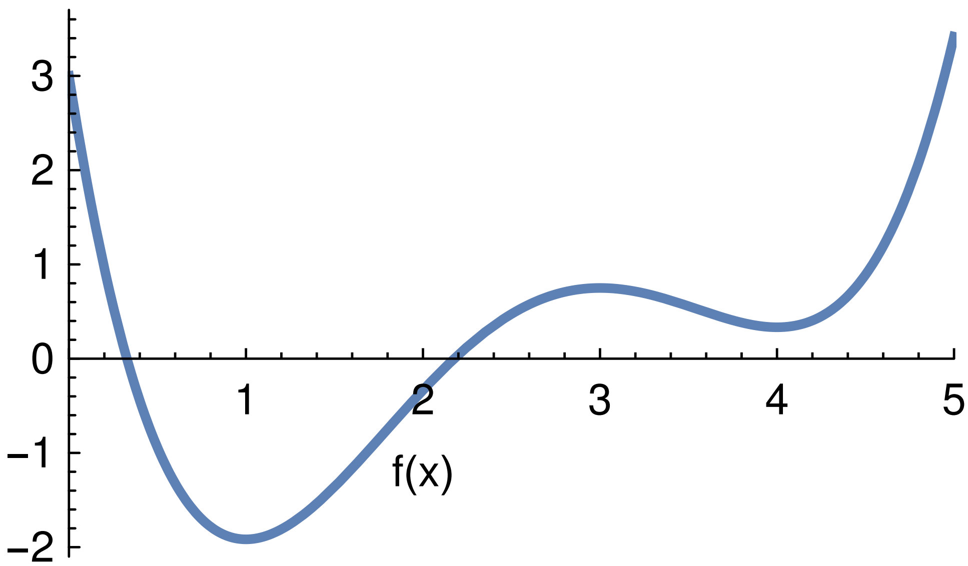
If $f''(a) > 0$, then $f'(x)$ is increasing at $x = a$.

(second-derivative rule)

- If $f''(a) > 0$, then $f(x)$ is concave up at $x = a$.
- If $f''(a) < 0$, then $f(x)$ is concave down at $x = a$.
- If $f''(a) = 0$, then $f(x)$ might have an inflection point at $x = a$.

An **inflection point** is a point, where concavity is changing.

[From concave up to down, or the other way around.]



Example 4. Approximately, on which intervals is $f(x)$ concave up/down?

(a) $f(x)$ is concave up for: $x < 1.8$ and $x > 3.5$

(b) $f(x)$ is concave down for: $1.8 < x < 3.5$

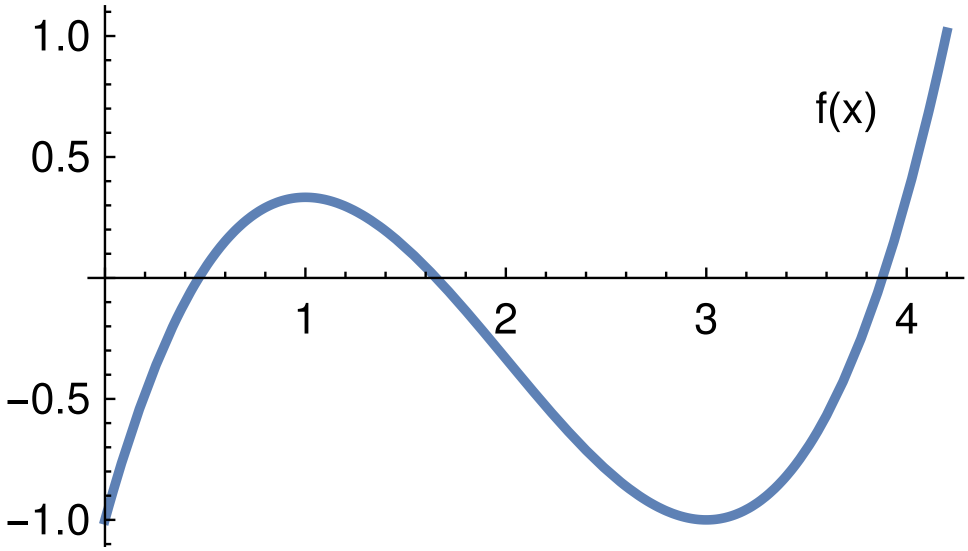
(c) $f(x)$ has inflection points at: $x \approx 1.8$ and $x \approx 3.5$

Here is a visual way to think of concavity and inflection points:

Imagine yourself riding a bike along the graph of $f(x)$. If the graph is a straight line, then you are steering neither left nor right. Usually, however, the graph is curved and you will have to steer either a little left or a little right.

Steering left means the graph is concave up (at that point), steering right means the graph is concave down. An inflection point is a point where you are transitioning from steering one direction to the other.

3 Finding local extrema



Example 5. Approximately, describe $f(x)$. What are the implications for $f'(x)$ and $f''(x)$?

(a)	increasing for: $x<1$ and $x>3$	$f'>0$
(b)	decreasing for: $1<x<3$	$f'<0$
(c)	local extrema: local max at $x=1$, local min at $x=3$	$f'=0$
(d)	concave up for: $x>2$	$f''>0$
(e)	concave down for: $x<2$	$f''<0$
(f)	inflection points: at $x\approx 2$	$f''=0$

Recall. If $f(x)$ has a local extremum at $x=a$, then $f'(a)=0$

[or $f'(a)$ does not exist]

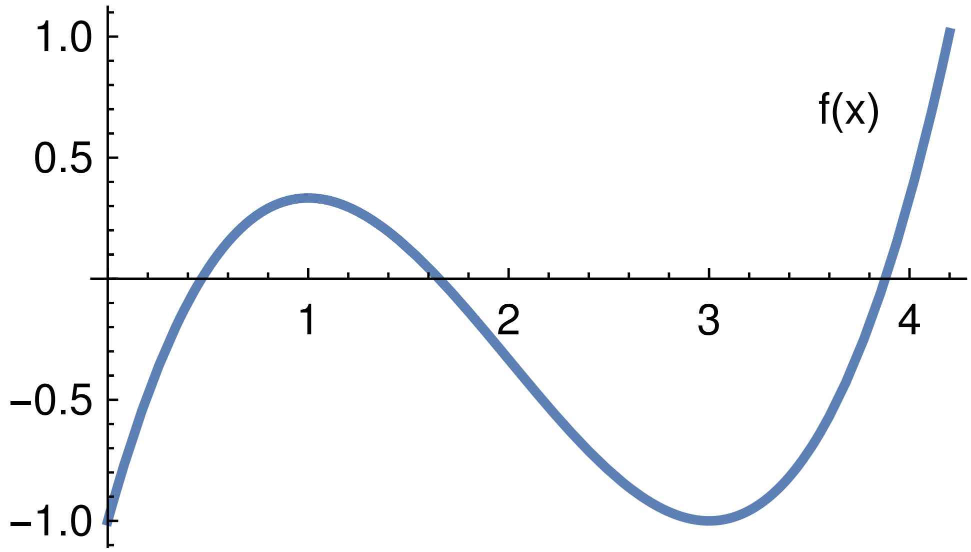
Why? Because $f'(a)>0$ would mean the function is increasing at $x=a$.

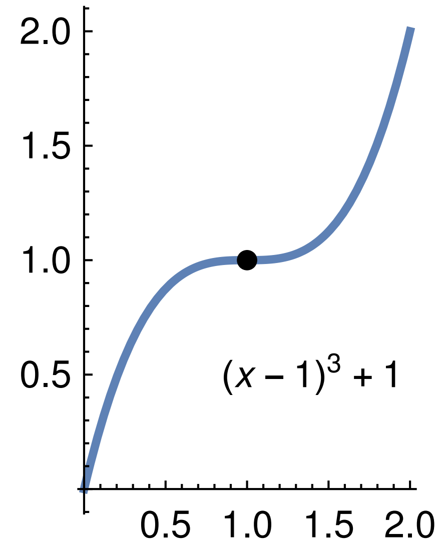
And, $f'(a)<0$ would mean the function is decreasing at $x=a$. In either case, there cannot be a local extremum at $x=a$.

To find extrema, we solve $f'(x)=0$ for x.

Such x are called **critical values**.

Not all critical values are extrema (see the plot of $(x-1)^3+1$ below).





3.1 First-derivative test

Observation.

- At a local max, f changes from increasing to decreasing.
- At a local min, f changes from decreasing to increasing.

(first-derivative test) Suppose $f'(a)=0$.

- If $f'(x)$ changes from positive to negative at $x=a$, then $f(x)$ has a local max at $x=a$.
- If $f'(x)$ changes from negative to positive at $x=a$, then $f(x)$ has a local min at $x=a$.

Example 6. Find the local extrema of $f(x)=\frac{1}{3}x^3-2x^2+3x-1$.

Solution. We use the first-derivative test.

$f'(x)=x^2-4x+3$

Solving $f'(x)=0$ (that's a quadratic equation) we find $x=1$ and $x=3$.

intervals	$x<1$	$x=1$	$1<x<3$	$x=3$	$x>3$	
$f'(x)$	$+$	0	$-$	0	$+$	we can determine the sign by computing $f'(x)$ for some x in the interval
$f(x)$	$\nearrow$		$\searrow$		$\nearrow$	

Hence, $f(x)$ has a local maximum at $x=1$.

And $f(x)$ has a local minimum at $x=3$.

[See Example 1 in Section 2.3 for more words.]

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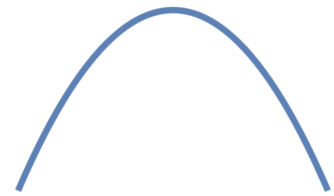
3.2 Second-derivative test

Observation.



- At a local max, we expect f to be concave down.
- At a local min, we expect f to be concave up.

(second-derivative test) Suppose $f'(a) = 0$.

- If $f''(a) < 0$, then $f(x)$ has a local max at $x = a$.
- If $f''(a) > 0$, then $f(x)$ has a local min at $x = a$.

Example 7. (again, alternative solution)

Find the local extrema of $f(x) = \frac{1}{3}x^3 - 2x^2 + 3x - 1$.

Solution. We use the second-derivative test.

$f'(x) = x^2 - 4x + 3$

Solving $f'(x) = 0$ (that's a quadratic equation) we find $x = 1$ and $x = 3$.

We compute the second derivative $f''(x) = 2x - 4$.

Since $f''(1) = -2 < 0$, $f(x)$ has a local max at $x = 1$.

Since $f''(3) = 2 > 0$, $f(x)$ has a local min at $x = 3$.

When to use which test?

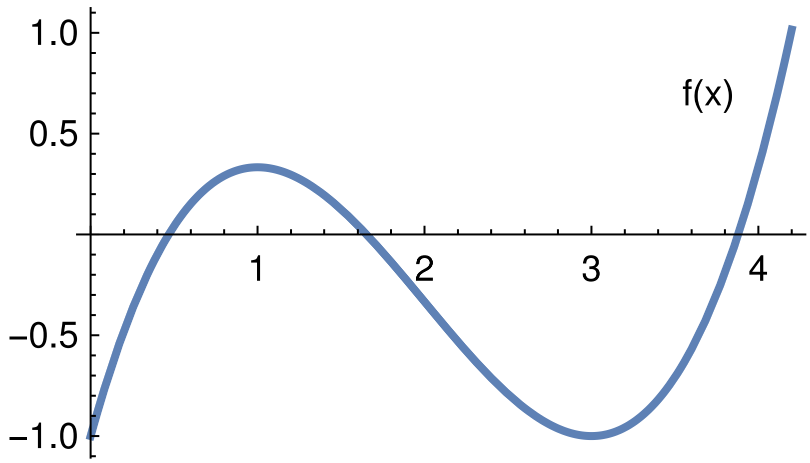
Rule of thumb: If $f''(a)$ is easy to compute, use the second-derivative test.

Otherwise, or if $f''(a) = 0$, use the first-derivative test.

[If you have computed all a such that $f'(x) = 0$, then the first-derivative test is easy to apply because we can quickly determine the sign of $f'(x)$ for any x.]

Geometric meaning of first and second derivative.

- $f'(a) > 0 \quad \Longrightarrow \quad f(x)$ is increasing at $x = a$
- $f'(a) < 0 \quad \Longrightarrow \quad f(x)$ is decreasing at $x = a$
- $f''(a) > 0 \quad \Longrightarrow \quad f(x)$ is concave up at $x = a$
- $f''(a) < 0 \quad \Longrightarrow \quad f(x)$ is concave down at $x = a$



Determine the sign $(+/-/0)$:

- $f'(0.5) > 0$ (increasing)
  $f''(0.5) < 0$ (concave down)
- $f'(1) = 0$
  $f''(1) < 0$ (concave down)
- $f'(4) > 0$ (increasing)
  $f''(4) > 0$ (concave up)

Inflection point: at $x \approx 2$

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