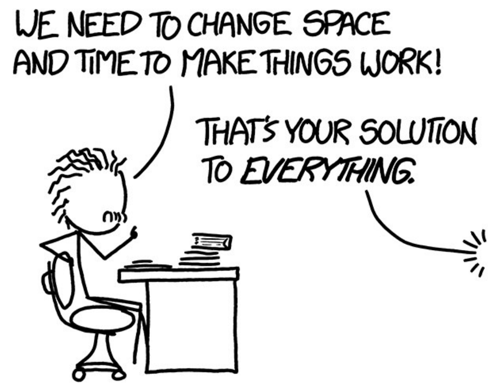
"**Compound interest** is the most powerful force in the universe."

often attributed to Albert Einstein
(urban legend?)



https://www.newyorker.com/tech/elements/the-space-doctors-big-idea-einstein-general-relativity

Q. Recently, GE stock lost 10%. What if it gains 10% today?

(a) It would be worth less than before.
(b) It would be worth the same as before.
(c) It would be worth more than before.

Less! Suppose $100\$$ initially.

After losing 10%, $90\$$ left. Gaining 10%, lifts price to $99\$$.

Important.

- Losing 10% means price gets multiplied with 0.9,
- gaining 10% means price gets multiplied with 1.1.
- Overall: $0.9 \cdot 1.1 = 0.99$, i.e., 1% lost.

Q. If your investment has an annual return of 5%, how long before it has doubled in value?

(a) About 10 years.
(b) About 15 years.
(c) About 20 years.
(d) About 25 years.

About 15 years! (Exactly 20 years without compounding.)

Each year, investment gets multiplied with 1.05.

After n years, multiplied with 1.05^n. When is $1.05^n = 2$?

$n = \log_{1.05} 2 \approx 14.2$

After 15 years, $1.05^{15} \approx 2.079$ (i.e. more than doubled).

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1 Interest

3y CD: 1.98% interest rate, 2.00% APY (GS Bank, 11/2/2017)

https://www.gsbank.com/savings-products/high-yield-cds.html

Ever wondered? What's the difference?

- Interest is paid monthly. Each month, $\frac{1}{12} \cdot 1.98\%$.

  That is, money is multiplied with $1 + \frac{1}{12} \cdot \frac{1.98}{100}$ each month.

- After a year, $\left(1 + \frac{1}{12} \cdot \frac{1.98}{100}\right)^{12} \approx 1.0200$.

  Hence the APY (annual percentage yield) of 2.00%.

Example 1. Consider the following two options:

(a) You receive 5% interest at the end of the year.

(b) You receive $\frac{5}{12}\%$ interest at the end of each month.

Which is better? How much total annual interest in second case?

Solution. Because of compounding interest, (b) is better.

$\left(1 + \frac{1}{12} \cdot \frac{5}{100}\right)^{12} \approx 1.05116$

Annual interest is about 5.116%.

Example 2. What about $\frac{5}{365}\%$ interest at the end of each day?

Solution. $\left(1 + \frac{1}{365} \cdot \frac{5}{100}\right)^{365} \approx 1.05127$

Annual interest is about 5.127%.

In the limit: (interest at the end of each hour/minute/second...)

Annual interest is $e^{5/100} \approx 5.127\%$. [insignificantly better than daily!]

Again, $e \approx 2.718$ shows up "naturally"!

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2 Interest compounded continously

Continous compounding: $P(t) = P_0\, e^{r t}$

P_0 initial amount; $P(t)$ amount after time t; r interest rate

Example 3. 1000 USD in savings with 2% interest compounded continuously.

(a) What is the balance $A(t)$ after t years?

(b) What differential equation is satisfied by $A(t)$?

(c) How much money will be in the account after 3 years?

(d) When will the balance reach 2000 USD?

(e) How fast is the balance growing when it reaches 2000 USD?

Solution.

(a) $A(t) = 1000 \cdot e^{0.02t}$

(b) $A'(t) = 1000 \cdot e^{0.02t} \cdot 0.02 = 0.02 \cdot A(t)$

Hence, $A(t)$ satisfies the DE $A'(t) = 0.02 \cdot A(t)$.

(c) $A(3) = 1000 \cdot e^{0.02 \cdot 3} \approx 1061.84$ USD

(d) Need to solve $A(t) = 2000$. That is, $1000 \cdot e^{0.02t} = 2000$.

Hence, $e^{0.02t} = 2$. So, $0.02t = \ln(2)$ and $t = \frac{\ln(2)}{0.02} \approx 34.66$.

The balance will reach 2000 USD after $t \approx 34.66$ years.

(e) This is asking for $A'(34.66)$.

Already computed: $A'(t) = 1000 \cdot e^{0.02t} \cdot 0.02$

$A'(34.66) = 1000 \cdot e^{0.02 \cdot 34.66} \cdot 0.02 = 40.00$

That is, the balance is growing at a rate of 40 USD/year.

Why such a nice answer?

$A'(t) = 0.02 \cdot A(t)$ and $0.02 \cdot 2000 = 40$

(again) Continous compounding: $P(t) = P_0\, e^{r t}$

P_0 initial amount; $P(t)$ amount after time t; r interest rate

Note that $P'(t) = r \cdot P(t)$. That's a differential equation.

That is, rate of change of P is proportional to P.

Example 4. Growth of population size $P(t)$ (insects, cells, ...) is often modelled in the exact same way. See 5.1 homework!

Model: rate of change of $P(t)$ is proportional to population size $P(t)$.

Reasonable in the absence of resource limitations, predators, ...

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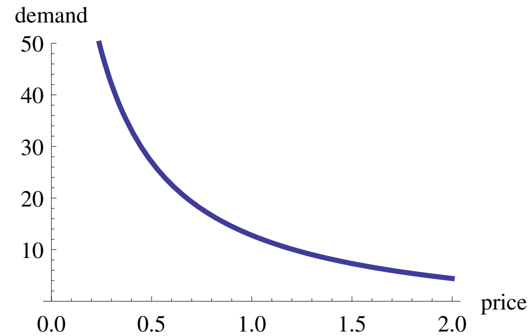
3 Elasticity of demand (application)

p price and q quantity ("demand") depend on each other.

Typical demand function $q = f(p)$ is decreasing.

Why? Higher price p, lower demand q.



Revenue is $R(p) = p \cdot f(p)$. Its rate of change is:

$$\begin{aligned} R'(p) &= 1 \cdot f(p) + p \cdot f'(p) \qquad \text{(product rule)} \\ &= f(p) \cdot \left[1 + \frac{p f'(p)}{f(p)}\right] \\ &= f(p) \cdot [1 - E(p)] \end{aligned}$$

The quantity $E(p) = -\frac{p f'(p)}{f(p)}$ is called **elasticity of demand**.

Why minus? Almost always, $f' < 0$. The minus makes elasticity positive.

$$R'(p) = f(p) \cdot [1 - E(p)] \quad \text{with} \quad E(p) = -\frac{p f'(p)}{f(p)}$$

[The quantities $p, f(p), E(p)$ are >0.]

If $E(p) = 1$, then $R'(p) = 0$ (no change in revenue).

If $E(p) > 1$, then $R'(p) < 0$.

Increasing price p, means a decrease in revenue $R(p)$.

Decreasing price p, means an increase in revenue $R(p)$.

In the case $E(p) > 1$, we say **demand is elastic** (at price p).

Similarly, **inelastic** if $E(p) < 1$. Spell out the consequences!

Example 5. Consider the demand function $q = 50 - p^2$.

(a) Determine $E(p)$.

(b) Is demand elastic or inelastic at $p = 5$?

(c) If $p = 5$, how would a decrease in price affect revenue?

(d) At which p is demand elastic?

Solution.

(a) $E(p) = -\frac{p f'(p)}{f(p)}$. Here, $q = f(p) = 50 - p^2$. So, $f'(p) = -2p$.

$$E(p) = \frac{2p^2}{50 - p^2}$$

(b) $E(5) = \frac{2 \cdot 25}{50 - 25} = \frac{50}{25} = 2 > 1$ means demand is elastic at $p = 5$.

(c) Since demand is elastic, a decrease in price increases revenue.

(d) We solve $E(p) = \frac{2p^2}{50 - p^2} = 1$ (recall that >1 means elastic).

$$2p^2 = 50 - p^2 \quad \Longrightarrow \quad 3p^2 = 50 \quad \Longrightarrow \quad p = \sqrt{\frac{50}{3}} \approx 4.082$$

Demand is elastic at prices $p > 4.082$.

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Assignments.

- check out Sections 5.1, 5.2, 5.3 in the book
- do "5.1. Exponential growth and decay" (3 questions)
- do "5.2. Compound interest" (5 questions)
- do "5.3. Applications of ln(x) to economics" (3 questions)
- take "chapter 5 quiz" (5 questions)

  All five questions are taken from homework assignments.