1 Reminders

- In-class exam in two weeks (Sept 28/29)
  - pen and paper; no calculator; no notes
  - practice problems are already posted
  - online MLP test on same topics: take before 10/4
- Assignments are posted after every class
  - to be completed before next class
  - due date on MLP includes grace period; don't wait that long!
- Lecture is for "big picture"
  - with only one class/week, we don't always cover all details
  - most importantly: make sure every assignment makes sense
    — "Question Help": step-by-step, worked example, video
    — Piazza for extra questions
  - read along in the book ("eText" on MyLabsPlus)

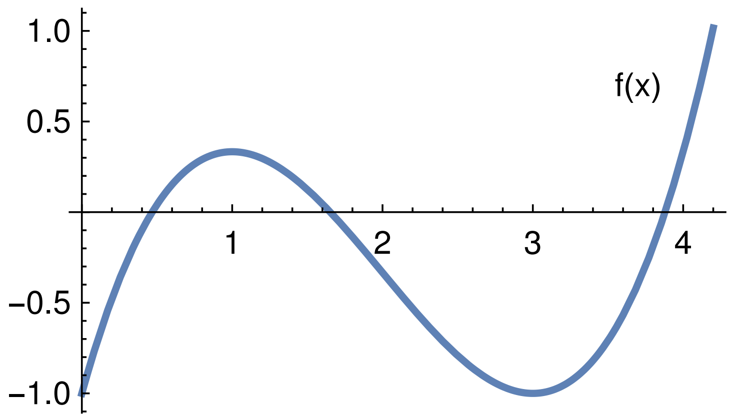
Missed an assignment/quiz? Hurricane worries, technical issues, sleepiness, ...?

Until the in-class exam, you are able to submit all of these late.

Please aim for 100% on the homework!

Geometric meaning of first and second derivative.

- $f'(a) > 0 \implies f(x)$ is increasing at $x = a$
- $f'(a) < 0 \implies f(x)$ is decreasing at $x = a$
- $f''(a) > 0 \implies f(x)$ is concave up at $x = a$
- $f''(a) < 0 \implies f(x)$ is concave down at $x = a$



Determine the sign $(+/-/0)$:

- $f'(0.5) > 0$ (increasing)
  $f''(0.5) < 0$ (concave down)
- $f'(1) = 0$
  $f''(1) < 0$ (concave down)
- $f'(4) > 0$ (increasing)
  $f''(4) > 0$ (concave up)

Inflection point: at $x \approx 2$

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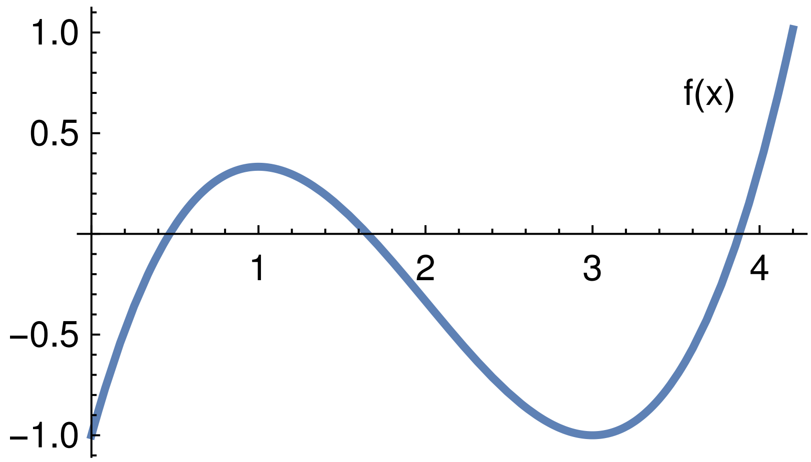
2 Finding local extrema

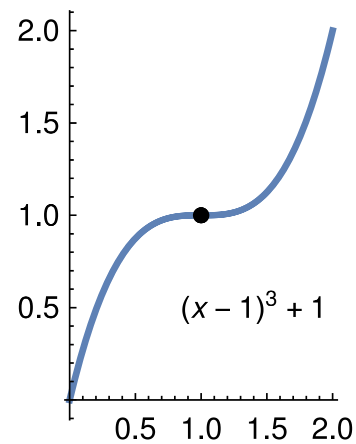
2.1 First-derivative test

To find extrema, we solve $f'(x) = 0$ for x.

Such x are called **critical values**.

Not all critical values are extrema (see the plot of $(x-1)^3 + 1$ below).





Observation.

- At a local max, f changes from increasing to decreasing.
- At a local min, f changes from decreasing to increasing.

(first-derivative test) Suppose $f'(a) = 0$.

- If $f'(x)$ changes from positive to negative at $x = a$, then $f(x)$ has a local max at $x = a$.
- If $f'(x)$ changes from negative to positive at $x = a$, then $f(x)$ has a local min at $x = a$.

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Example 1. Find the local extrema of $f(x) = \frac{1}{3}x^3 - 2x^2 + 3x - 1$.

Solution. We use the first-derivative test.

$f'(x) = x^2 - 4x + 3$

Solving $f'(x) = 0$ (that's a quadratic equation) we find $x = 1$ and $x = 3$.

intervals	$x < 1$	$x = 1$	$1 < x < 3$	$x = 3$	$x > 3$	
$f'(x)$	$+$	0	$-$	0	$+$	we can determine the sign by computing $f'(x)$ for some x in the interval
$f(x)$	$\nearrow$		$\searrow$		$\nearrow$	

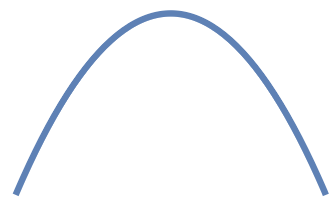
Hence, $f(x)$ has a local maximum at $x = 1$.

And $f(x)$ has a local minimum at $x = 3$.

[See Example 1 in Section 2.3 for more words.]

2.2 Second-derivative test

Observation.



- At a local max, we expect f to be concave down.
- At a local min, we expect f to be concave up.

(second-derivative test) Suppose $f'(a) = 0$.

- If $f''(a) < 0$, then $f(x)$ has a local max at $x = a$.
- If $f''(a) > 0$, then $f(x)$ has a local min at $x = a$.

Example 2. (again, alternative solution)

Find the local extrema of $f(x) = \frac{1}{3}x^3 - 2x^2 + 3x - 1$.

Solution. We use the second-derivative test.

$f'(x) = x^2 - 4x + 3$

Solving $f'(x) = 0$ (that's a quadratic equation) we find $x = 1$ and $x = 3$.

We compute the second derivative $f''(x) = 2x - 4$.

Since $f''(1) = -2 < 0$, $f(x)$ has a local max at $x = 1$.

Since $f''(3) = 2 > 0$, $f(x)$ has a local min at $x = 3$.

When to use which test?

Rule of thumb: If $f''(a)$ is easy to compute, use the second-derivative test.

Otherwise, or if $f''(a) = 0$, use the first-derivative test.

[If you have computed all a such that $f'(x) = 0$, then the first-derivative test is easy to apply because we can quickly determine the sign of $f'(x)$ for any x.]

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Example 3. Given the cost function $C(x) = x^3 - 12x^2 + 60x + 20$, find the minimal marginal cost.

Solution.

The marginal cost function is $M(x) = C'(x) = 3x^2 - 24x + 60$.

We need to find the minimum of $M(x)$.

$M'(x) = 6x - 24$

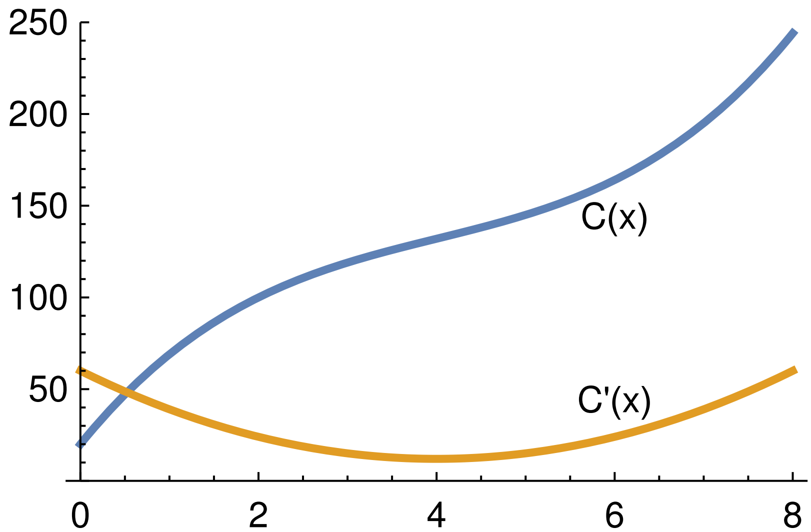
Solving $M'(x) = 0$, we find $x = 4$.

Let us check that this is a minimum:

[You could skip this step by arguing that $x = 4$ must be the minimum because it is the only candidate.]

- (second derivative test) $M''(x) = 6$
  Since $M''(4) = 6 > 0$, this is a local minimum.
- Because there is no other critical points, this must be the absolute minimum.

The minimal marginal cost is $M(4) = 12$.



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Example 4. At a price of 6 dollars, 50 beers were sold per night at a local cinema. When the price was raised to 7 dollars, sales dropped to 40 beers.

(a) Assuming a linear demand curve, which price maximizes revenue?

Solution.

(setup) p prize per beer, x number of beer sold

Revenue is $R(x) = p \cdot x$. **(objective equation)**

Linear demand means that $p = ax + b$ (a line!) for some a, b.

We know that $(x_1, p_1) = (50, 6)$ and $(x_2, p_2) = (40, 7)$.

Hence, $p - 6 = \underbrace{\text{slope}}_{\frac{p_2 - p_1}{x_2 - x_1} = \frac{1}{-10}} (x - 50)$.

This simplifies to $p = 11 - \frac{1}{10}x$. **(constraint equation)**

Revenue is $R(x) = p \cdot x = \left(11 - \frac{1}{10}x\right) \cdot x = 11x - \frac{1}{10}x^2$.

(find max of $R(x)$) $R'(x) = 11 - \frac{2}{10}x$

Solving $R'(x) = 11 - \frac{1}{5}x = 0$, we find $x = 55$.

The corresponding price is $p = 11 - \frac{1}{10} \cdot 55 = 5.5$ dollars.

(Then the revenue is $R(55) = 5.5 \cdot 55 = 302.5$ dollars.)

(b) Suppose the cinema has fixed costs of 100 dollars per night for selling beer, and variable costs of 2 dollars per beer. Find the price that maximizes profit.

Solution.

(setup) p prize per beer, x number of beer sold

Cost is $C(x) = 100 + 2x$.

As before, revenue is $R(x) = 11x - \frac{1}{10}x^2$.

Profit is $P(x) = R(x) - C(x) = 9x - \frac{1}{10}x^2 - 100$.

(find max of $P(x)$) $P'(x) = 9 - \frac{2}{10}x$

Solving $P'(x) = 9 - \frac{1}{5}x = 0$, we find $x = 45$.

The corresponding price is $p = 11 - \frac{1}{10} \cdot 45 = 6.5$ dollars.

(Then the profit is $P(45) = 102.5$ dollars.)

Assignments.

- check out Sections 2.3, 2.5, 2.7 in the book
- finish "2.3. First and second derivative tests" (5 questions)
- take "2.2, 2.3. Graphing quiz" (4 questions)

  This quiz is more tricky than others. Below is some information on what to expect.
- do "2.5. Optimization" (3 questions)
- do "2.7. Applications to Business and Economics" (4 questions)
- if you feel ready, take "2.5, 2.7 Optimization quiz" (3 questions)

  All three questions are taken from the homework assignments.

4 Graphing quiz

Quiz #3 consists of 4 questions:

- Given the signs of $f'(x)$ and $f''(x)$, determine where the relative extrema and inflection points of $f(x)$ are.

  This problem is similar to Problem 4 on the In-Class Exam Prep. Do that first!

  Review. $f(x)$ has a local extremum at $x=a$ if f' is changing sign at $x=a$.

  - f' changing from $+$ to $-$ $\Longrightarrow$ local max
  - f' changing from $-$ to $+$ $\Longrightarrow$ local min

  Review. $f(x)$ has an inflection point at $x=a$ if f'' is changing sign at $x=a$.

- Find the equation of a tangent line for $f(x)$ by reading off the slope from a graph of $f'(x)$.
- Determine relative extreme point of a quadratic function. Then, decide if it is a min or max.

  When MyLabsPlus asks for an extreme **point**, you must give both coordinates. That is, if the extremum is at $x=a$, you must enter $(a, f(a))$.

- Given a plot of $f'(x)$, make a statement about $f(x)$.

  From a plot of $f'(x)$, how can you tell where $f(x)$ is increasing/decreasing? How can you tell where the local extrema of $f(x)$ are? [See first question.]

  Then, given a plot of $f''(x)$, make a statement about $f(x)$.

  From a plot of $f''(x)$, how can you tell where $f(x)$ is concave up/down? How can you tell where the inflection points of $f(x)$ are? [See first question.]