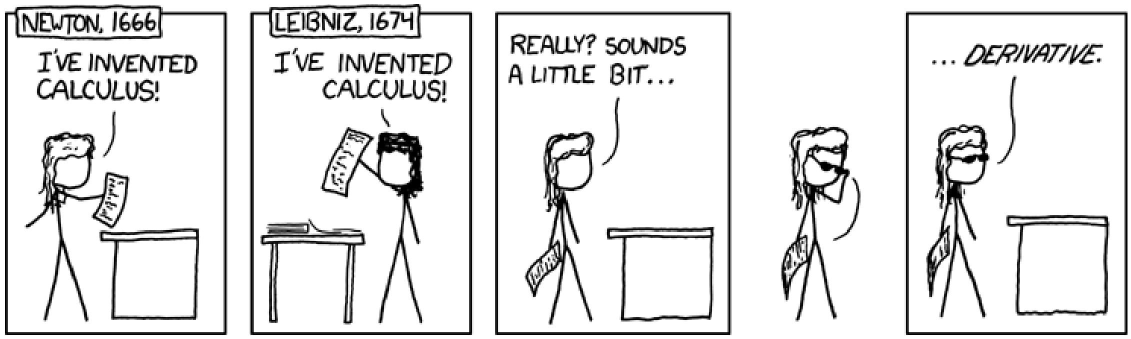
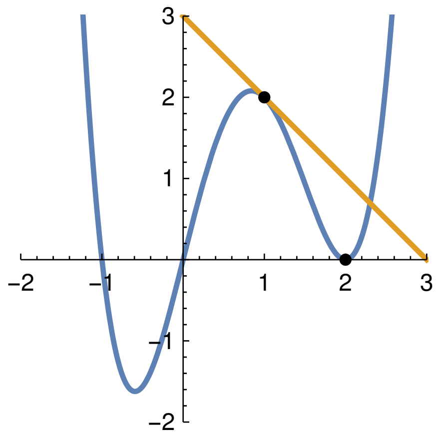


https://xkcd.com/626/

With $f(x)$ as in the graph, estimate:



$f(1) = 2$

$f'(1) = -1$

$f(2) = 0$

$f'(2) = 0$

If $f(x) = x^4 - 3x^3 + 4x$, (that's the function in the plot)

then $f'(x) = 4x^3 - 9x^2 + 4$.

In particular, $f'(1) = 4 - 9 + 4 = -1$ and $f'(2) = 4 \cdot 8 - 9 \cdot 4 + 4 = 0$.

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1 Rate of change

$f'(a)$ is the

- **slope of the tangent line** approximating $f(x)$ at $x = a$
- **rate of change** of $f(x)$ at $x = a$

Recall. $\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x}$ $\left(\text{i.e. } \frac{\text{change in } y}{\text{change in } x}\right)$

This also explains why we write $\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$ if $y = f(x)$.

Example 1. Suppose your fresh cup of coffee is $f(t)$ degrees (Fahrenheit) warm after t minutes.

(a) What is the meaning of $f(5) = 175$?

First off, the units for $f(5)$ are degrees.

Meaning: After 5 minutes, your coffee is 175 degrees warm.

(b) What is the meaning of $f'(5) = -2$?

First off, the units for $f'(5)$ are degrees/min.

Meaning: after 5 minutes (at that moment of time), your coffee is cooling down 2 degrees/minute.

[This is the rate at which the temperature changes.]

(c) Estimate the temperature after 6 minutes.

In other words, estimate $f(6)$.

At $t = 5$, the temperature is $f(5) = 175$ degrees, and
it changes at a rate of $f'(5) = -2$ degrees/minute.

Hence, we estimate $f(6) \approx 175 - 2 = 173$ degrees.

Note. Mathematically, we have approximated $f(t)$ with the tangent line at $t = 5$ (which has equation $f(5) + f'(5)(t - 5)$).

(d) Estimate the temperature after 8 minutes.

As before, we now estimate $f(8) \approx 175 - 2 \cdot 3 = 169$.

This estimate is more risky since 8 is further away from 5.

Fancy thoughts. Should we expect $f(8) < 169$ or $f(8) > 169$?

The rate of change should decrease as the coffee approaches room temperature. Hence, we expect that $f(8) > 169$ and that $f'(8) > -2$.

Comment. We might discuss Newton's law of cooling when talking about exponential models.

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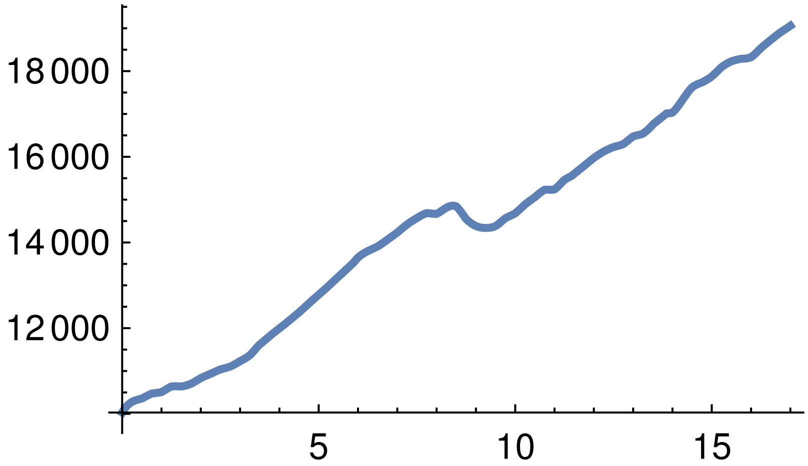
Example 2. Let $g(t)$ be the U.S. GDP in billions of dollars at time t in years since Jan 1, 2000.



(a) What is measured by $g'(t)$?

First off, the units for $g'(t)$ are billion dollars/year.

$g'(t)$ is the change in GDP in billion dollars/year at time t.

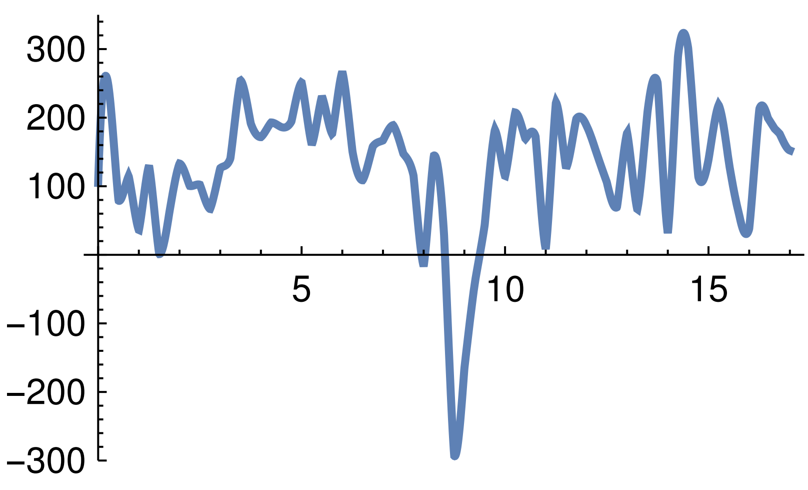
(b) When is $g'(t) < 0$?

An exact answer is hard to read off the graph.

However, $g'(t)$ is mostly positive, with a notable exception around $t = 9$, when $g'(t) < 0$ (the 2009 recession).

Below is an approximation to $g'(t)$.

(The data was available only quarterly. Also, we should consider the possibility that $g(t)$ is not differentiable; for instance, stock prices jump so erratically that the graph does not admit tangent lines.)



Data from FRED (Federal Reserve Bank of St. Louis); retrieved Aug 2017
https://fred.stlouisfed.org/series/GDP

2 Marginal cost/revenue/profit

- If $C(x)$ is the cost to produce x units, then
- $C'(x)$ is the **marginal cost** (at production level x).

Marginal cost is measured in cost/unit.
It is the cost per (additional) unit at production level x.
Note that $C'(x) \approx \frac{C(x+1) - C(x)}{1}$.
The right-hand side is literally the cost to produce one more item. However, it is beneficial to also allow fractional units, in which case $C'(x)$ is more appropriate.

Example 3. Suppose the cost (in dollars) of producing x units of a product is given by $C(x) = \mathrm{secret}(x)$ dollars.

(a) What is the cost of producing 50 units?

$C(50)$ dollars

(b) What is the marginal cost when the production level is 50 units?

$C'(50)$ dollars/unit

(c) At what level of production, is the marginal cost 100 dollars/unit?

Need to solve $C'(x) = 100$.
Each such x is a level of production when the marginal cost is 100 dollars/unit.
(There could be several such levels x of production.)

(d) How many units can we produce with 1000 dollars?

Need to solve $C(x) = 1000$.
Then, x is the number of units can we produce with 1000 dollars.

Profit is revenue minus cost: $P(x) = R(x) - C(x)$.

As before, x is the production level.

Marginal revenue and marginal profit are likewise defined:

- **Marginal revenue** is $R'(x)$.

  This is the (extra) revenue for an additional unit (at production level x).

- **Marginal profit** is $P'(x)$.

  This is the (extra) profit for an additional unit (at production level x).

3 Next stop: pies!

Angela: So, wait, when pies are involved, you can suddenly do math in your head?

Oscar: Hold on, Kevin, how much is 19,154 pies divided by 61 pies?

Kevin: 314 pies.

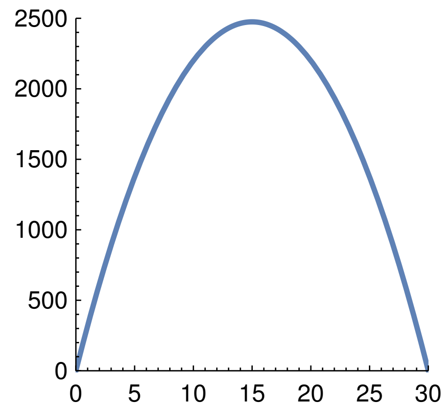
Oscar: What if it were salads?

Kevin: Well, it's the...carry the four...and...it doesn't work.

The Office (Season 9, Episode 4):
http://www.simplethingcalledlife.com/stcl/when-pies-are-involved/

Any comments on Kevin's answer?

Example 4. Suppose $s(t)$ is the height in miles after t minutes of a rocket that is shot up vertically.



(a) What is the meaning of $s(5) = 1375$?

First off, units: $s(5)$ is miles.

After 5 minutes, the rocket is 1375 miles high.

(b) What is the meaning of $s'(5) = 220$?

First off, units: $s'(5)$ is miles/min.

After 5 minutes, the rocket has a speed of 220 miles/min (13200 miles/h).

(c) What is the meaning of $s''(5) = -22$?

First off, units: $s''(5)$ is (miles/min)/min, or miles/min^2.

After 5 minutes, the rocket has an acceleration of -22 miles/min^2.

Physics comment. Earth's gravitation is about 22 miles/min^2 (or 32.2 ft/sec^2).

In other words, our rocket is ballistic (only initially powered, then in free fall).

(d) When is the altitude of the rocket 2000 miles?

To find such a time t, we need to solve $s(t) = 2000$.

[The picture suggests $t \approx 8.5$ and $t \approx 21.5$.]

(e) When does the rocket land again?

To find that time, we need to solve $s(t) = 0$.

One solution is $t = 0$ but we are looking for the other one.

[The picture suggests $t = 30$.]

(f) What is the maximal height the rocket reaches?

To find the time t of maximal height, we need to solve $s'(t) = 0$.

[The picture suggests $t = 15$ and a maximal height of $s(15) \approx 2500$ miles.]

Just for fun. These numbers are all made up. However, they are (in some aspects) not too far off from the 2017/7/28 launch of a North Korea missile. That missile reached a height of about 2315 miles and landed after 47 minutes.

https://en.wikipedia.org/wiki/Hwasong-14

For comparison, the ISS is 205-270 miles above earth, the moon $238,900$ miles.

Example 5. Solve the last three parts of the previous problem if $s(t) = 330t - 11t^2$.

(a) When is the altitude of the rocket 2000 miles?

To find such a time t, we need to solve $s(t) = 2000$.

$330t - 11t^2 = 2000$, that is, $-11t^2 + 330t - 2000 = 0$

has the two solutions $t = \frac{-330 \pm \sqrt{330^2 - 4(-11)(-2000)}}{-22} = 8.429, 21.571$.

(b) When does the rocket land again?

To find that time, we need to solve $s(t) = 0$.

$\underbrace{330t - 11t^2}_{= t(330 - 11t)} = 0$ has the solutions $t = 0$ and $t = \frac{330}{11} = 30$.

As suggested by the graph, the rocket lands at $t = 30$.

(c) What is the maximal height the rocket reaches?

To find the time t of maximal height, we need to solve $s'(t) = 0$.

$s'(t) = 330 - 22t = 0$ has the solution $t = \frac{330}{22} = 15$.

Thus, the maximal height is $s(15) = 2475$ miles.

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Assignments.

- finish "1.7. More on derivatives" (8 questions)
- check out Section 1.8 in the book
- do "1.8. Rate of change" (4 questions)
- take "chapter 1 quiz" (10 questions)

4 Our second quiz

Keep in mind that you can take each quiz a second time if you are unhappy with your first score.

The second quiz has 10 questions, covering the following:

- given slope and point of line, complete point-slope equation
- estimate slope of tangent line from picture
- power rule (2x)
- derivative of polynomial (2x)
- first and second derivative
- evaluate a derivative at some value, like $\frac{\mathrm{d}}{\mathrm{d}x}(5x-2)^{10}\Big|_{x=2}$
- rate of change text problem
- velocity, acceleration