In-class Exam #1: Prep

MATH 120 — Calculus & Applications
September 28/29

Please print your name:

No notes, calculators or tools of any kind are permitted. There are 22 points in total.

Good luck!

The actual in-class exam will be similar but shorter (with more space for answers).

Problem 1. (2 points) Given $f(x) = 2x^4 - 3\sqrt{x} + 7x - 4^2$, compute $f'(x)$.

Solution. $f'(x) = 8x^3 - \frac{3}{2}x^{-1/2} + 7$ □

Problem 2. (2 points) Consider the graph of $y = 1 + \sqrt{x}$. Determine the tangent line at $x = 4$.

Solution. Since $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}x^{-1/2}$, the slope is $\frac{\mathrm{d}y}{\mathrm{d}x}\Big|_{x=4} = \frac{1}{2} \cdot 4^{-1/2} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

If $x = 4$ then $y = 1 + \sqrt{4} = 3$, so the tangent line passes through the point $(4, 3)$.

Therefore, the tangent line is $y - 3 = \frac{1}{4}(x - 4)$.

[Optionally, in slope-intercept form, this is $y = \frac{1}{4}x + 2$.] □

Problem 3. (2 points) Consider the function $f(x) = 2x^3 + 5x$.

(a) Is $f(x)$ increasing/decreasing at $x = -1$?

(b) Is $f(x)$ concave up/down at $x = -1$?

Solution.

(a) $f'(x) = 6x^2 + 5$

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$f'(-1) = 11 > 0$

Hence, $f(x)$ is increasing at $x = -1$.

(b) $f''(x) = 12x$

$f'(-1) = -12 < 0$

Hence, $f(x)$ is concave down at $x = -1$. □

Problem 4. (3 points) The first and second derivatives of the function $f(x)$ have the following values:

	$x<-2$	$x=-2$	$-2<x<-1$	$x=-1$	$-1<x<0$	$x=0$	$0<x<1$	$x=1$	$1<x<3$	$x=3$	$x>3$
$f'(x)$	$-$	0	$+$	$+$	$+$	0	$+$	$+$	$+$	0	$-$
$f''(x)$	$+$	$+$	$+$	0	$-$	0	$+$	0	$-$	0	$-$

Determine the location of all local minima, local maxima and inflection points.

Solution. In summary, we have a local min at $x = -2$, a local max at $x = 3$, and inflection points at $x = -1$, $x = 0$, $x = 1$.

The reasoning is as follows:

Local extrema can only occur when $f'(x) = 0$. Hence, the candidates are $x = -2$, $x = 0$ and $x = 3$. If f' is changing from $+$ to $-$, then we have a local max. Likewise, if f' is changing from $-$ to $+$, then we have a local min.

- At $x = -2$: since f' is changing from $-$ to $+$, there is a local min at $x = -2$.

  (Alternatively, we could have noticed that $f''(-2) > 0$, which implies that this is a local min.)

- At $x = 0$: since the sign of f' is not changing, we do not have a local extremum at $x = 0$.

  (Since $f''(0) = 0$, the second-derivative test would not help us decide whether this is a local extremum or not.)

- At $x = 3$: since f' is changing from $+$ to $-$, there is a local max at $x = 3$.

  (Since $f''(0) = 0$, the second-derivative test would not help us decide whether this is a local extremum or not.)

Inflection points can only occur when $f''(x) = 0$. Hence, the candidates are $x = -1$, $x = 0$, $x = 1$ and $x = 3$. Recall that $f(x)$ has an inflection point at $x = a$ if f'' is changing sign at $x = a$ (i.e. concavity is changing).

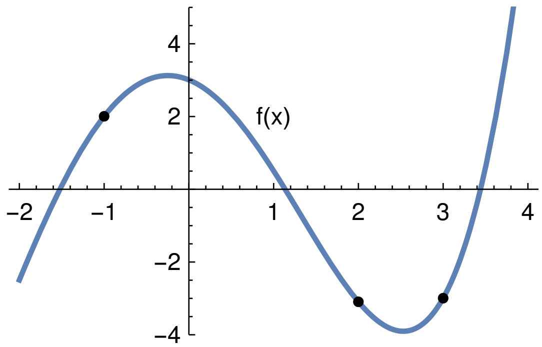
- At $x = -1$: since f'' is changing from $+$ to $-$, there is an inflection point at $x = -1$.

- At $x = 0$: since f'' is changing from $-$ to $+$, there is an inflection point at $x = 0$.

- At $x = 1$: since f'' is changing from $+$ to $-$, there is an inflection point at $x = 1$.

- At $x = 3$: since the sign of f'' is not changing (f is concave down before and after), we do not have an inflection point at $x = 3$. □

Problem 5. **(3 points)** Use the graph below to fill in each entry of the grid with positive, negative or zero.



	$f(x)$	$f'(x)$	$f''(x)$
$x=-1$	$+$	$+$	$-$
$x=2$	$-$	$-$	$+$
$x=3$	$-$	$+$	$+$

Problem 6. **(2 points)** A classmate needs to find the local extrema of the function $f(x)=x^4-\frac{4}{3}x^3-4x^2+24x+1$. She already found that the critical points are at $x=-1$, $x=0$ and $x=2$. Help her conclude what the local extrema are.

Solution. We will use the second-derivative test.

$f'(x)=4x^3-4x^2-8x+24$

$f''(x)=12x^2-8x-8$

Since $f''(-1)=12+8-8=12>0$, $f(x)$ has a local min at $x=-1$.

Since $f''(0)=-8<0$, $f(x)$ has a local max at $x=0$.

Since $f''(2)=48-16-8=24>0$, $f(x)$ has a local min at $x=2$.

Alternative. Since we have a complete list of critical points (i.e. there is no other x for which $f'(x)=0$), we can also use the first-derivative test. However, since the second derivative is so easy to compute, the second-derivative test should be our first choice. □

Problem 7. **(2 points)** Let $T(x)$ be the time in hours it takes to produce x units.

(a) The units for $T'(x)$ are hours/unit.

(b) The units for $T''(x)$ are (hours/unit)/unit = hours/unit^2.

Problem 8. **(3 points)** A small rectangular garden of area 80 square meters is to be surrounded on three sides by a brick wall costing 5 dollars per meter and on one side by a fence costing 3 dollars per meter. Find the dimensions of the garden such that the cost of the fence is minimized.

Solution. Let a be the length in meters of the side with a fence, and b the length of the other side.

Then, the cost for the fence is $C=(5+3)a+(5+5)b=8a+10b$. (This is the objective function.)

On the other hand, we have $ab=80$. (This is a constraint equation.)

In order to minimize the cost, we express cost as a function of a. Since $b=\frac{80}{a}$ (because $ab=80$), we get that the cost is $C(a)=8a+10\cdot\frac{80}{a}=8a+800a^{-1}$.

$C'(a)=8+800\cdot(-a^{-2})=8-800a^{-2}$.

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We now solve $C'(a) = 0$ to find the critical values: $8 - 800a^{-2} = 0$ simplifies to $a^2 - 100 = 0$ (divide both sides by 8 and multiply with a^2), that is, $a^2 = 100$. Therefore, $a = \sqrt{100} = 10$ (the other solution is $a = -10$ but a negative length does not make sense here).

Now, we could use the first or second-derivative test to determine that $a = 10$ is a local minimum. Since there are no other critical points, it then follows that $a = 10$ is in fact the absolute minimum. Alternatively, observe that for small a (close to 0) and large a, the cost is definitely not optimal (actually the cost becomes arbitrarily large); hence, the absolute minimum must be somewhere in between, and the only candidate is $a = 10$ (this observation makes the first or second-derivative test unnecessary). As a third alternative, make a quick sketch of $C(a)$.

If $a = 10$, then $b = \frac{80}{a} = 8$.

In conclusion, to minimize costs, the length of the side with a fence should be 10 meters and the length of the other side should be 8 meters.

Comment. We could also have expressed the cost as a function of b. Then $C(b) = 8 \cdot \frac{80}{b} + 10b = 640b^{-1} + 10b$ and $C'(b) = -640b^{-2} + 10$, so that $C'(b) = 0$ simplifies to $b^2 = 64$. We would conclude that $b = 8$ and then determine $a = \frac{80}{b} = 10$, ending up (of course!) with the same dimensions as before. □

Problem 9. **(3 points)** Given the cost function $C(x) = \frac{1}{2}x^3 - 15x^2 + 200x + 4$, find the minimal marginal cost.

Solution. The marginal cost function is $M(x) = C'(x) = \frac{3}{2}x^2 - 30x + 200$.

We need to find the minimum of $M(x)$.

$M'(x) = 3x - 30$

Solving $M'(x) = 0$, that is, $3x - 30 = 0$, we find $x = 10$.

Since $M''(10) = 3 > 0$, this is a local minimum. Since there is no other critical points, this must be the absolute minimum.

The minimal marginal cost is $M(10) = \frac{3}{2} \cdot 100 - 300 + 200 = 50$. □