Practice for Midterm #3

MATH 125 — Calculus 1
Tuesday, April 16

Please print your name:

Besides the allowed calculator, no notes or tools of any kind will be permitted.

- Have another look at the homework sets #10, #11, #12, #13, especially those problems that you struggled with.
- Retake Quizzes 7 and 8! (Versions with and without solutions are posted to our course website.)
- Go through the lecture sketches (posted to our course website) and do the problems we did in class (ignore the solutions until you have solved the problem yourself).

Problem 1. Compute the following limits.

(a) $\lim_{x\to\infty} x e^{-2x}$

(b) $\lim_{x\to\infty} \frac{\sqrt{x}}{\ln(x^4+1)}$

(c) $\lim_{x\to 0} \frac{\cos(2x)-1}{\sin(3x)}$

(d) $\lim_{x\to 0} \frac{\cos(2x)-1}{\sin(3x^2)}$

(e) $\lim_{x\to\infty} \left(1-\frac{3}{x^2}\right)^x$

(f) $\lim_{x\to 1} \left(\frac{1}{\ln(x)}-\frac{1}{x-1}\right)$

Solution.

(a) $\lim_{x\to\infty} x e^{-2x} = \lim_{x\to\infty} \frac{x}{e^{2x}} \underset{\text{“}\frac{\infty}{\infty}\text{”}}{\overset{\text{LH}}{=}} \lim_{x\to\infty} \frac{1}{2e^{2x}} = 0$

(b) $\lim_{x\to\infty} \frac{\sqrt{x}}{\ln(x^4+1)} \underset{\text{“}\frac{\infty}{\infty}\text{”}}{\overset{\text{LH}}{=}} \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{\frac{4x^3}{x^4+1}} = \lim_{x\to\infty} \frac{x^4+1}{8x^{7/2}} = \lim_{x\to\infty} \frac{x^{1/2}+x^{-7/2}}{8} = \infty$

(c) $\lim_{x\to 0} \frac{\cos(2x)-1}{\sin(3x)} \underset{\text{“}\frac{0}{0}\text{”}}{\overset{\text{LH}}{=}} \lim_{x\to 0} \frac{-2\sin(2x)}{3\cos(3x)} = \frac{-2\cdot 0}{3\cdot 1} = 0$

(d) $\lim_{x\to 0} \frac{\cos(2x)-1}{\sin(3x^2)} \underset{\text{“}\frac{0}{0}\text{”}}{\overset{\text{LH}}{=}} \lim_{x\to 0} \frac{-2\sin(2x)}{6x\cos(3x^2)} \underset{\text{“}\frac{0}{0}\text{”}}{\overset{\text{LH}}{=}} \lim_{x\to 0} \frac{-4\cos(2x)}{6\cos(3x^2)-36x^2\sin(3x^2)} = \frac{-4\cdot 1}{6\cdot 1-0} = -\frac{2}{3}$

(e) This limit is of the form “1^∞”. To apply L'Hospital, we first note that

$$\ln\left(\left(1-\frac{3}{x^2}\right)^x\right) = x\ln\left(1-\frac{3}{x^2}\right) = \frac{\ln\left(1-\frac{3}{x^2}\right)}{x^{-1}}$$

and that, as $x\to\infty$, the RHS results in a limit of the form “$\frac{0}{0}$”. Since

$$\lim_{x\to\infty} \frac{\ln\left(1-\frac{3}{x^2}\right)}{x^{-1}} \underset{\text{“}\frac{0}{0}\text{”}}{\overset{\text{LH}}{=}} \lim_{x\to\infty} \frac{\frac{6x^{-3}}{1-\frac{3}{x^2}}}{-x^{-2}} = \lim_{x\to\infty} \frac{-6x^{-1}}{1-\frac{3}{x^2}} = \lim_{x\to\infty} \frac{-6}{x-\frac{3}{x}} = 0,$$

$\lim_{x\to\infty} \left(1-\frac{3}{x^2}\right)^x = e^0 = 1.$

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(f) $\lim_{x\to 1}\left(\frac{1}{\ln(x)}-\frac{1}{x-1}\right)=\lim_{x\to 1}\frac{x-1-\ln(x)}{(x-1)\ln(x)}\overset{\text{LH}}{\underset{\text{“}\frac{0}{0}\text{”}}{=}}\lim_{x\to 1}\frac{1-\frac{1}{x}}{\ln(x)+\frac{x-1}{x}}\overset{\text{LH}}{\underset{\text{“}\frac{0}{0}\text{”}}{=}}\lim_{x\to 1}\frac{\frac{1}{x^2}}{\frac{1}{x}+\frac{1}{x^2}}=\frac{1}{1+1}=\frac{1}{2}$ □

Problem 2. Evaluate the following indefinite integrals.

(a) $\int (x^4+3x^2-7x+1)\mathrm{d}x$

(b) $\int \left(\sqrt{x}-\frac{1}{x^4}\right)\mathrm{d}x$

(c) $\int (e^{2x}-7e^{-3x})\mathrm{d}x$

(d) $\int \frac{\sin(4x)+2\cos(3x)}{5}\mathrm{d}x$

Solution.

(a) $\int (x^4+3x^2-7x+1)\mathrm{d}x=\frac{1}{5}x^5+x^3-\frac{7}{2}x^2+x+C$

(b) $\int \left(\sqrt{x}-\frac{1}{x^4}\right)\mathrm{d}x=\frac{2}{3}x^{3/2}+\frac{1}{3x^3}+C$

(c) $\int (e^{2x}-7e^{-3x})\mathrm{d}x=\frac{1}{2}e^{2x}+\frac{7}{3}e^{-3x}+C$

(d) $\int \frac{\sin(4x)+2\cos(3x)}{5}\mathrm{d}x=\frac{1}{5}\left(-\frac{1}{4}\cos(4x)+\frac{2}{3}\sin(3x)\right)+C$ □

Problem 3. Evaluate the following definite integrals.

(a) $\int_2^3 (x^2-2)\mathrm{d}x$

(b) $\int_1^4 \left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)\mathrm{d}x$

(c) $\int_0^3 e^{4x}\mathrm{d}x$

(d) $\int_1^2 \left(\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}\right)\mathrm{d}x$

(e) $\int_0^{\pi/2} \sin(2x)\mathrm{d}x$

Solution.

(a) $\int_2^3 (x^2-2)\mathrm{d}x=\left[\frac{1}{3}x^3-2x\right]_2^3=3-\left(-\frac{4}{3}\right)=\frac{13}{3}$

(b) $\int_1^4 \left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)\mathrm{d}x=\int_1^4 (x^{1/2}-x^{-1/2})\mathrm{d}x=\left[\frac{2}{3}x^{3/2}-2x^{1/2}\right]_1^4=\frac{4}{3}-\left(-\frac{4}{3}\right)=\frac{8}{3}$

(c) $\int_0^3 e^{4x}\mathrm{d}x=\left[\frac{1}{4}e^{4x}\right]_0^3=\frac{1}{4}e^{12}-\frac{1}{4}$

(d) $\int_1^2 \left(\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}\right)\mathrm{d}x=\left[\ln(x)-\frac{1}{x}-\frac{1}{2x^2}\right]_1^2=\left(\ln(2)-\frac{5}{8}\right)-\left(\ln(1)-\frac{3}{2}\right)=\ln(2)+\frac{7}{8}$

(e) $\int_0^{\pi/2} \sin(2x)\mathrm{d}x=\left[-\frac{1}{2}\cos(2x)\right]_0^{\pi/2}=-\frac{1}{2}\cos(\pi)-\left(-\frac{1}{2}\cos(0)\right)=1$ □

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Problem 4. Suppose you have 160 m of fencing and want to fence off a rectangular field that borders a straight river (no fence is needed alongside the river). What is the maximum area you can fence off?

Solution. Let a and b be the lengths of the two sides (in m) of the field, with b the one along the river.

Then the required fencing is $2a+b$ and the area to be maximized is $A=a\cdot b$. Since $2a+b=160$ we find $b=160-2a$ and hence $A=a\cdot(160-2a)$. To find the maximum of A for a in $[0,80]$ we compute the critical points:

$$\frac{\mathrm{d}A}{\mathrm{d}a}=160-4a=0 \quad\Longrightarrow\quad a=40$$

Since the maximum clearly does not occur for the endpoints $a=0$ and $a=80$, the maximum must occur at the only critical point $a=40$. The corresponding maximum area is $40\cdot(160-2\cdot 40)=3200$ m^2. □

Problem 5. You are designing a rectangular poster to contain 50 in^2 of printing with a 4 in margin at the top and bottom, and a 2 in margin at each side. What overall dimensions will minimize the amount of paper used?

Solution. Suppose the printed part of the poster has height x (in) and width y (in).

Our goal is to minimize $A=(x+8)(y+4)$. The constraint is $xy=50$.

From the constraint, $y=\frac{50}{x}$, so we need to minimize $A=(x+8)\left(\frac{50}{x}+4\right)=82+4x+\frac{400}{x}$ (with x in $(0,\infty)$).

$\frac{\mathrm{d}A}{\mathrm{d}x}=4-\frac{400}{x^2}=0$ is equivalent to $x^2=100$, so that $x=\pm 10$, of which only $x=10$ is practical.

Since values of x close to 0 and ∞ (the endpoints) are clearly not minimizing A, the absolute minimum of A occurs at $x=10$ (the only critical point). The corresponding optimal overall dimensions for the poster are height $x+8=18$ in and width $y+4=9$ in. □

Problem 6. Evaluate the following sums.

(a) $\displaystyle\sum_{k=3}^{6} 2^{-k}(1-(-1)^k)$

(b) $\displaystyle\sum_{k=1}^{4} \frac{(-1)^k}{\sqrt{k}}$

Solution.

(a) $\displaystyle\sum_{k=3}^{6} 2^{-k}(1-(-1)^k)=2^{-3}\cdot 2+2^{-4}\cdot 0+2^{-5}\cdot 2+2^{-6}\cdot 0=\frac{1}{4}+\frac{1}{16}=\frac{5}{16}$

(b) $\displaystyle\sum_{k=1}^{4} \frac{(-1)^k}{\sqrt{k}}=-\frac{1}{1}+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}+\frac{1}{2}=\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}-\frac{1}{2}$ □

Problem 7. Let A be the (net) area between the x-axis and $f(x)$ for x in $[2,6]$.

(a) Write down a Riemann sum for A using 5 intervals (of equal size) and

  (1) left endpoints,

  (2) right endpoints,

  (3) midpoints.

(b) Using sigma notation, write down a Riemann sum for A using n intervals (of equal size) and

  (1) left endpoints,

  (2) right endpoints,

(3) midpoints.

Solution.

(a) Each interval has length $\frac{6-2}{5} = \frac{4}{5}$. The first interval is $\left[2, 2+\frac{4}{5}\right] = \left[2, \frac{14}{5}\right]$, which has midpoint $\frac{12}{5}$.

(1) The 5 left endpoints are 2, $2+\frac{4}{5} = \frac{14}{5}$, $\frac{18}{5}$, $\frac{22}{5}$, $\frac{26}{5}$ (each is $\frac{4}{5}$ after the previous).

The Riemann sum therefore is $\frac{4}{5}\left[f(2) + f\left(\frac{14}{5}\right) + f\left(\frac{18}{5}\right) + f\left(\frac{22}{5}\right) + f\left(\frac{26}{5}\right)\right]$.

(2) The 5 right endpoints are $\frac{14}{5}$, $\frac{18}{5}$, $\frac{22}{5}$, $\frac{26}{5}$, $\frac{30}{5} = 6$ (each is $\frac{4}{5}$ after the previous).

The Riemann sum therefore is $\frac{4}{5}\left[f\left(\frac{14}{5}\right) + f\left(\frac{18}{5}\right) + f\left(\frac{22}{5}\right) + f\left(\frac{26}{5}\right) + f(6)\right]$.

(3) The 5 midpoints are $\frac{12}{5}$, $\frac{16}{5}$, $\frac{20}{5} = 4$, $\frac{24}{5}$, $\frac{28}{5}$ (each is $\frac{4}{5}$ after the previous).

The Riemann sum therefore is $\frac{4}{5}\left[f\left(\frac{12}{5}\right) + f\left(\frac{16}{5}\right) + f(4) + f\left(\frac{24}{5}\right) + f\left(\frac{28}{5}\right)\right]$.

(b) Now, each interval has length $\frac{6-2}{n} = \frac{4}{n}$. The first interval is $\left[2, 2+\frac{4}{n}\right]$, which has midpoint $2+\frac{2}{n}$.

(1) The n left endpoints are 2, $2+\frac{4}{n}$, $2+2\cdot\frac{4}{n}$, $2+3\cdot\frac{4}{n}$, ... $2+(n-1)\cdot\frac{4}{n}$ (each is $\frac{4}{n}$ after the previous).

The Riemann sum therefore is $\sum_{k=0}^{n-1} \frac{4}{n} f\left(2+k\cdot\frac{4}{n}\right)$.

(2) The n right endpoints are $2+\frac{4}{n}$, $2+2\cdot\frac{4}{n}$, $2+3\cdot\frac{4}{n}$, ... $2+n\cdot\frac{4}{n} = 6$ (each is $\frac{4}{n}$ after the previous).

The Riemann sum therefore is $\sum_{k=1}^{n} \frac{4}{n} f\left(2+k\cdot\frac{4}{n}\right)$.

(3) The n midpoints are $2+\frac{2}{n}$, $2+\frac{2}{n}+\frac{4}{n}$, $2+\frac{2}{n}+2\cdot\frac{4}{n}$, ... $2+\frac{2}{n}+(n-1)\cdot\frac{4}{n}$ (each is $\frac{4}{n}$ after the previous).

The Riemann sum therefore is $\sum_{k=0}^{n-1} \frac{4}{n} f\left(2+\frac{2}{n}+k\cdot\frac{4}{n}\right)$. □

Problem 8. Let A be the area between the x-axis and $f(x) = \sqrt{x}$ for x in $[1, 5]$.

(a) Estimate the area A using a Riemann sum with 3 intervals and left endpoints.

(b) Estimate the area A using a Riemann sum with 3 intervals and midpoints.

(c) Compute the (exact) area A.

Solution.

(a) Each interval has length $\frac{5-1}{3} = \frac{4}{3}$. The first interval is $\left[1, 1+\frac{4}{3}\right] = \left[1, \frac{7}{3}\right]$.

The 3 left endpoints therefore are 1, $1+\frac{4}{3} = \frac{7}{3}$, $\frac{7}{3}+\frac{4}{3} = \frac{11}{3}$ (each is $\frac{4}{3}$ after the previous).

The estimate for A is

$$\frac{4}{3}\left[f(1) + f\left(\frac{7}{3}\right) + f\left(\frac{11}{3}\right)\right] = \frac{4}{3}\left[1 + \sqrt{\frac{7}{3}} + \sqrt{\frac{11}{3}}\right] \approx 5.923.$$

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[Since $f(x)$ is increasing, we know that this is an underestimate.]

(b) Each interval has length $\frac{5-1}{3} = \frac{4}{3}$. The first interval is $\left[1, 1 + \frac{4}{3}\right] = \left[1, \frac{7}{3}\right]$ and has midpoint $\frac{5}{3}$.

The 3 midpoints therefore are $\frac{5}{3}$, $\frac{5}{3} + \frac{4}{3} = 3$, $3 + \frac{4}{3} = \frac{13}{3}$ (each is $\frac{4}{3}$ after the previous).

The estimate for A is

$$\frac{4}{3}\left[f\left(\frac{5}{3}\right) + f(3) + f\left(\frac{13}{3}\right)\right] = \frac{4}{3}\left[\sqrt{\frac{5}{3}} + \sqrt{3} + \sqrt{\frac{13}{3}}\right] \approx 6.806.$$

(c) $A = \int_1^5 \sqrt{x}\mathrm{d}x = \left[\frac{2}{3}x^{3/2}\right]_1^5 = \frac{2}{3} \cdot 5^{3/2} - \frac{2}{3} = \frac{10}{3}\sqrt{5} - \frac{2}{3}$

[For comparison, $\frac{10}{3}\sqrt{5} - \frac{2}{3} \approx 6.787$.] □

Problem 9.

(a) Estimate the average value of $f(x) = \frac{1}{x^2}$ on $[1, 3]$ using a Riemann sum with 3 intervals and midpoints.

(b) Compute the (exact) average value of $f(x) = \frac{1}{x^2}$ on $[1, 3]$.

Solution.

(a) Each interval has length $\frac{3-1}{3} = \frac{2}{3}$. The first interval is $\left[1, 1 + \frac{2}{3}\right] = \left[1, \frac{5}{3}\right]$ and has midpoint $\frac{4}{3}$.

The 3 midpoints therefore are $\frac{4}{3}$, $\frac{4}{3} + \frac{2}{3} = 2$, $2 + \frac{2}{3} = \frac{8}{3}$ (each is $\frac{2}{3}$ after the previous).

The estimate for the average value is

$$\frac{1}{3-1}\left(\frac{2}{3}f\left(\frac{4}{3}\right) + \frac{2}{3}f(2) + \frac{2}{3}f\left(\frac{8}{3}\right)\right) = \frac{1}{3}\left(f\left(\frac{4}{3}\right) + f(2) + f\left(\frac{8}{3}\right)\right) = \frac{1}{3}\left(\frac{9}{16} + \frac{1}{4} + \frac{9}{64}\right) = \frac{61}{192}.$$

(b) The average value is $\frac{1}{3-1}\int_1^3 \frac{1}{x^2}\mathrm{d}x = \frac{1}{2}\left[-\frac{1}{x}\right]_1^3 = \frac{1}{2}\left(-\frac{1}{3} - (-1)\right) = \frac{1}{3}$.

[For comparison, our estimate was $\frac{61}{192} \approx 0.3177$.] □