Midterm #2

MATH 125 — Calculus 1
Tuesday, March 12

Please print your name:

Besides the allowed calculator, no notes or tools of any kind are permitted. There are 27 points in total.

Good luck!

Problem 1. (5 points) Compute the following derivatives. [No need to show work here.]

(a) $\frac{d}{dx}[x^7 - 2x^3 + e^2] =$

(b) $\frac{d}{dx}\frac{1}{\sqrt{x}} =$

(c) $\frac{d}{dx}\ln(x^3 - 1) =$

(d) $\frac{d}{dx}[x^3 \tan^{-1}(x)] =$

(e) $\frac{d}{dx}e^{\cos(4x)} =$

Solution.

(a) $\frac{d}{dx}[x^7 - 2x^3 + e^2] = 7x^6 - 6x^2$

(b) $\frac{d}{dx}\frac{1}{\sqrt{x}} = -\frac{1}{2}x^{-3/2}$

(c) $\frac{d}{dx}\ln(x^3 - 1) = \frac{3x^2}{x^3 - 1}$

(d) $\frac{d}{dx}[x^3 \tan^{-1}(x)] = 3x^2 \tan^{-1}(x) + \frac{x^3}{x^2 + 1}$

(e) $\frac{d}{dx}e^{\cos(4x)} = -4\sin(4x)e^{\cos(4x)}$ □

Problem 2. (1 point) By the limit definition, $f'(7) =$.

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Solution. $f'(7) = \lim_{h \to 0} \frac{f(7+h) - f(7)}{h}$ □

Problem 3. (2 points) Compute $\frac{\mathrm{d}}{\mathrm{d}x}(x+4)^x$.

Solution. We apply logarithmic differentiation: Let $y = (x+4)^x$. Then $\ln(y) = x\ln(x+4)$. Differentiating both sides, we obtain

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \ln(x+4) + \frac{x}{x+4}.$$

Solving for $\frac{\mathrm{d}y}{\mathrm{d}x}$, we find $\frac{\mathrm{d}y}{\mathrm{d}x} = (x+4)^x\left[\ln(x+4) + \frac{x}{x+4}\right]$. □

Problem 4. (3+1+1 points) Consider the curve $x^2 + xy = e^y$. [Show your work!]

(a) Using implicit differentiation, determine $\frac{\mathrm{d}y}{\mathrm{d}x}$.

(b) Determine the line tangent to the curve at the point $(-1, 0)$.

(c) Determine the line normal to the curve at the point $(-1, 0)$.

Solution.

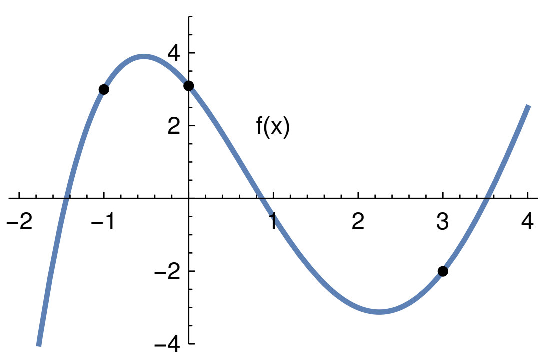
(a) Applying $\frac{\mathrm{d}}{\mathrm{d}x}$ to both sides of $x^2 + xy = e^y$, we obtain $2x + y + x\frac{\mathrm{d}y}{\mathrm{d}x} = e^y\frac{\mathrm{d}y}{\mathrm{d}x}$, so that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x+y}{e^y - x}$.

(b) The slope of the line tangent to the curve at $(-1, 0)$ is $\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]_{x=-1, y=0} = \left[\frac{2x+y}{e^y - x}\right]_{x=-1, y=0} = \frac{-2+0}{1+1} = -1$.

Hence, the tangent line has equation $y = -1(x+1)$ or, equivalently, $y = -x - 1$.

(c) The normal line has slope $-\frac{1}{-1} = 1$ and, hence, equation $y = x + 1$. □

Problem 5. (3 points) Use the graph below to fill in each entry of the grid with positive, negative or zero.



	$f(x)$	$f'(x)$	$f''(x)$
$x=-1$	$+$	$+$	$-$
$x=0$	$+$	$-$	$-$
$x=3$	$-$	$+$	$+$

Problem 6. (2 points) Roughly sketch a differentiable function $f(x)$ with the following property.

(a) $f'(0) = 0$ but 0 is not a local extremum,

(b) $f'(0) < 0$ and $f''(0) > 0$.

Solution.

(a) You can sketch $f(x) = x^3$.

(b) You can sketch $f(x) = (x-1)^2$ or any other function which is decreasing at $x = 0$ and concave up. □

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Problem 7. (3+1+1+1 points) Consider the function $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 1$.

(a) Determine all local extrema of $f(x)$.

(b) On which (open) intervals is $f(x)$ increasing?

(c) On which (open) intervals is $f(x)$ concave up?

(d) $f(x)$ has an inflection point at $x =$ ____.

Solution.

(a) Because the derivatives of $f(x)$ are pleasant to compute, we will use the second-derivative test.

Since $f'(x) = x^2 - x - 2 = (x+1)(x-2)$, the critical points are at $x = -1$ and $x = 2$.

$f''(x) = 2x - 1$

Since $f''(2) = 3 > 0$, $f(x)$ has a local min at $x = 2$.

Since $f''(-1) = -3 < 0$, $f(x)$ has a local max at $x = -1$.

(b) $f(x)$ is increasing on $(-\infty, -1)$ and $(2, \infty)$.

(c) Solving $f''(x) > 0$, we find that $f(x)$ is concave up on $\left(\frac{1}{2}, \infty\right)$.

(d) Solving $f''(x) = 0$, we find that $f(x)$ has an inflection point at $x = \frac{1}{2}$. (We know that there must be an inflection point between the local max (concave down) and the local min (concave up).) □

Problem 8. (3 points) Oil is leaking from a tanker and spreads in a circle whose area increases at a rate of 10 km^2/h. How fast is the radius of the spill increasing after 3 h?

Solution. Let A be the area (in km^2) and r the radius (in km) of the circular spill. Then A and r are related by the equation $A = \pi r^2$. It follows that the rates of change, with respect to time t (in h), are related by

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 2\pi r \frac{\mathrm{d}r}{\mathrm{d}t}.$$

We have $\frac{\mathrm{d}A}{\mathrm{d}t} = 10$. After $t = 3$, the area is $A = 3 \cdot 10$, so that the radius is $r = \sqrt{\frac{A}{\pi}} = \sqrt{\frac{30}{\pi}}$. It follows that

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{2\pi r}\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{10}{2\pi\sqrt{\frac{30}{\pi}}} = \sqrt{\frac{5}{6\pi}} \approx 0.515 \text{ km/h}.$$ □

(extra scratch paper)