Midterm #1

MATH 125 — Calculus 1
Tuesday, Feb 5

Please print your name:

Besides the allowed calculator, no notes or tools of any kind are permitted. There are 25 points in total.

Good luck!

Problem 1. (6 points) Determine the following limits (or state that they don't exist). [No need to show work here.]

(a) $\lim_{x \to 0} \frac{\sin(3x)}{7x} =$ ☐

(b) $\lim_{x \to \infty} \frac{\sin(3x)}{7x} =$ ☐

(c) $\lim_{x \to 1} \frac{\sin(3x)}{7x} =$ ☐

(d) $\lim_{x \to \infty} \frac{\sqrt{x} + 7x^2 - 2}{3x^2 + 5} =$ ☐

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(e) If $\lim_{x\to a} f(x)=3$ and $\lim_{x\to a} g(x)=5$, then $\lim_{x\to a}[f(x)^2-3g(x)]=$ ☐.

(f) If $\lim_{x\to 1} f(x)=2$, $\lim_{x\to 1} g(x)=3$ and $\lim_{x\to 2} g(x)=4$, then $\lim_{x\to 1} g(f(x))=$ ☐.

Solution.

(a) $\lim_{x\to 0}\frac{\sin(3x)}{7x}=\frac{3}{7}$

(b) $\lim_{x\to \infty}\frac{\sin(3x)}{7x}=0$

(c) $\lim_{x\to 1}\frac{\sin(3x)}{7x}=\frac{\sin(3)}{7}$

(d) $\lim_{x\to \infty}\frac{\sqrt{x}+7x^2-2}{3x^2+5}=\frac{7}{3}$

(e) If $\lim_{x\to a} f(x)=3$ and $\lim_{x\to a} g(x)=5$, then $\lim_{x\to a}[f(x)^2-3g(x)]=3^2-3\cdot 5=-6$.

(f) Note that $f(x)\to 2$ as $x\to 1$, so that $\lim_{x\to 1} g(f(x))=\lim_{y\to 2} g(y)=4$. ☐

Problem 2. (2 points) Simplify! $e^{2\ln(x)-\ln(3y)}=$ ☐

Solution. $e^{2\ln(x)-\ln(3y)}=e^{\ln\left(\frac{x^2}{3y}\right)}=\frac{x^2}{3y}$ ☐

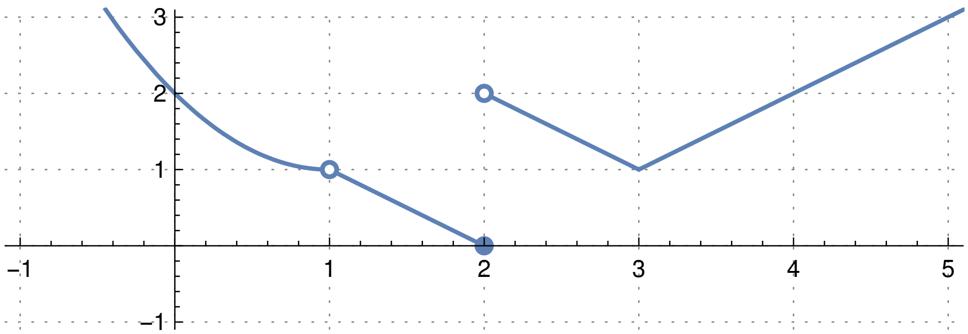
Problem 3. (2 points) Let $f(x)$ be a complicated continuous function taking the following values:

x	-3	-2	-1	0	1	2	3
$f(x)$	2	3	1	-1	-3	4	4

How many solutions to the equation $f(x)=3$ can we guarantee? ☐

Solution. By the intermediate value theorem, there must be a solution x to $f(x)=3$ in the interval $[1,2]$. We also know that $x=-2$ is a solution. We can therefore guarantee 2 solutions. ☐

Problem 4. (3 points) Let $f(x)$ be the function graphed below. Fill in the blanks.



(a) $f(x)$ is continuous everywhere except at the following values of x: [ ]

(b) $f(x)$ has a removable discontinuity at the following values of x: [ ]

(c) $\lim_{x\to 2^+} f(x) =$ [ ]

Solution.

(a) $f(x)$ is continuous everywhere except at $x = 1$ and $x = 2$.

(b) $f(x)$ has a removable discontinuity at $x = 1$.

(c) $\lim_{x\to 2^+} f(x) = 2$ □

Problem 5. (4 points) For what values of a is $f(x) = \begin{cases} x^2 - a, & x < 1, \\ a\ln(x) + 2, & x \geqslant 1, \end{cases}$ continuous at every x? [Show work!]

Solution. Observe that $f(x)$ is always continuous at every point except, possibly, $x = 1$.

In order for $f(x)$ to be continuous at $x = 1$, we need $\lim_{x\to 1} f(x) = f(1)$.

- $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2 - a) = 1 - a$
- $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (a\ln(x) + 2) = a\ln(1) + 2 = 2 = f(1)$

Hence, $\lim_{x\to 1} f(x) = f(1)$ if and only if $1 - a = 2$, which happens if and only if $a = -1$.

Thus, $f(x)$ is continuous if and only if $a = -1$. □

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Problem 6. (4 points) Determine $\lim_{x\to\infty}\left(x-\sqrt{x^2+7x}\right)$. [Show work!]

Solution. We have

$$x-\sqrt{x^2+7x}=\frac{\left(x-\sqrt{x^2+7x}\right)\left(x+\sqrt{x^2+7x}\right)}{x+\sqrt{x^2+7x}}=\frac{-7x}{x+\sqrt{x^2+7x}}=\frac{-7}{1+\sqrt{1+\frac{7}{x}}}\overset{x\to\infty}{\longrightarrow}-\frac{7}{1+\sqrt{1+0}}=-\frac{7}{2}. \qquad \square$$

Problem 7. (4 points) Determine $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ for $f(x)=x^2+1$. [Show work!]

Solution. Since $f(x+h)=(x+h)^2+1=x^2+2hx+h^2+1$, we have

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x^2+2hx+h^2+1)-(x^2+1)}{h}=\lim_{h\to 0}\frac{2hx+h^2}{h}=\lim_{h\to 0}(2x+h)=2x. \qquad \square$$

(extra scratch paper)