Example 147.

(a) Find the net area between the x-axis and $f(x) = -x^2 - 2x$ for $-3 \leqslant x \leqslant 2$.

(b) Find the total area between the x-axis and $f(x) = -x^2 - 2x$ for $-3 \leqslant x \leqslant 2$.

Solution.

(a) The net area is $\int_{-3}^{2} (-x^2 - 2x)\mathrm{d}x = \left[-\frac{1}{3}x^3 - x^2\right]_{-3}^{2} = \left(-\frac{8}{3} - 4\right) - (9 - 9) = -\frac{20}{3}$.

(b) Since $f(x) = -x(x+2)$, we have $f(x) = 0$ if and only if $x = 0$ or $x - 2$. Accordingly, we split $[-3, 2]$ into $[-3, -2]$ (where $f(x) < 0$), $[-2, 0]$ (where $f(x) > 0$), and $[0, 2]$ (where $f(x) < 0$).

$$\begin{aligned}
\text{total area} = \int_{-3}^{2} |f(x)|\mathrm{d}x &= \int_{-3}^{-2} |f(x)|\mathrm{d}x + \int_{-2}^{0} |f(x)|\mathrm{d}x + \int_{0}^{2} |f(x)|\mathrm{d}x \\
&= -\int_{-3}^{-2} f(x)\mathrm{d}x + \int_{-2}^{0} f(x)\mathrm{d}x - \int_{0}^{2} f(x)\mathrm{d}x \\
&= -\left[-\frac{1}{3}x^3 - x^2\right]_{-3}^{-2} + \left[-\frac{1}{3}x^3 - x^2\right]_{-2}^{0} - \left[-\frac{1}{3}x^3 - x^2\right]_{0}^{2} \\
&= \frac{4}{3} + \frac{4}{3} + \frac{20}{3} = \frac{28}{3}
\end{aligned}$$

Example 148. Find the area of the region bounded by $y = 1 + \cos(x)$, $y = 2$ and $x = \pi$.

Solution. Make a sketch!

$$\text{area} = 2\pi - \int_{0}^{\pi} (1 + \cos(x))\mathrm{d}x = 2\pi - \Big[x + \sin(x)\Big]_{0}^{\pi} = 2\pi - \pi = \pi$$

Look at your sketch again! Can you see how the curve $y = 1 + \cos(x)$ cuts the rectangle into two equal pieces?

Example 149. Determine $\int_{0}^{1} \frac{1}{\sqrt{x}}\mathrm{d}x$ and $\int_{0}^{1} \frac{1}{x}\mathrm{d}x$.

Comment. Make a sketch! Note that the areas corresponding to the integrals have infinite height. Such integrals are called **improper**: here, the integrand has a singularity at an endpoint of integration.

When working carefully, our integrals should be interpreted as $\lim_{a \to 0} \int_{a}^{1} \frac{1}{\sqrt{x}}\mathrm{d}x$ and $\lim_{a \to 0} \int_{a}^{1} \frac{1}{x}\mathrm{d}x$.

Solution.

$$\int_{0}^{1} \frac{1}{\sqrt{x}}\mathrm{d}x = \Big[2\sqrt{x}\Big]_{0}^{1} = 2$$

$$\int_{0}^{1} \frac{1}{x}\mathrm{d}x = \Big[\ln(x)\Big]_{0}^{1} = \infty$$ (More precisely, $\lim_{a \to 0} \int_{a}^{1} \frac{1}{x}\mathrm{d}x = \lim_{a \to 0} [-\ln(a)] = \infty$.)

More on improper integrals in Calculus II :)

Armin Straub
straub@southalabama.edu