Theorem 144. (Fundamental Theorem of Calculus) Let $f(x)$ be continuous on $[a,b]$.

(1) Then $F(x) = \int_a^x f(t)\mathrm{d}t$ is an antiderivative of $f(x)$. In other words,

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\mathrm{d}t = f(x).$$

(2) If $F(x)$ is an antiderivative of $f(x)$, then

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a).$$

Comment. The "first part" is saying that first integrating and then differentiating doesn't do anything. Writing the "second part" as $\int_a^b F'(x)\mathrm{d}x = F(b) - F(a)$, we can interpret it as saying that first differentiating (which kills constants!) and then integrating doesn't do anything up to a constant. Taken together, derivatives and integrals are essentially inverse operations: one undoes the other.

Proof?

(1) Define $F(x) = \int_a^x f(t)\mathrm{d}t$. Then: Make a sketch!

$$\begin{aligned} F'(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} \\ &= \lim_{h\to 0}\frac{1}{h}\left[\int_a^{x+h} f(t)\mathrm{d}t - \int_a^x f(t)\mathrm{d}t\right] \\ &= \lim_{h\to 0}\underbrace{\frac{1}{h}\int_x^{x+h} f(t)\mathrm{d}t}_{\text{average of } f \text{ on } [x,x+h]} = f(x) \end{aligned}$$

(2) Suppose that F is an antiderivative of f. By the first part, any such antiderivative is of the form $F(x) = \int_a^x f(t)\mathrm{d}t + C$. It then follows that

$$F(b) - F(a) = \left(\int_a^b f(t)\mathrm{d}t + C\right) - \left(\int_a^a f(t)\mathrm{d}t + C\right) = \int_a^b f(t)\mathrm{d}t.$$

Example 145. Compute $\frac{\mathrm{d}}{\mathrm{d}x}\int_{\sqrt{x}}^{\cos(2x)} \sin(t^2)\mathrm{d}t$.

Solution. For any a, we have: $\frac{\mathrm{d}}{\mathrm{d}x}\int_{\sqrt{x}}^{\cos(2x)} \sin(t^2)\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}x}\left[\int_{\sqrt{x}}^{a}\sin(t^2)\mathrm{d}t + \int_a^{\cos(2x)}\sin(t^2)\mathrm{d}t\right]$
$= \frac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^{\cos(2x)}\sin(t^2)\mathrm{d}t - \int_a^{\sqrt{x}}\sin(t^2)\mathrm{d}t\right] = -2\sin(2x)\sin(\cos^2(2x)) - \frac{\sin(x)}{2\sqrt{x}}$

Armin Straub
straub@southalabama.edu

Example 146. Find the total area between the x-axis and $f(x) = -x^2 - 2x$ for $-3 \leqslant x \leqslant 2$.

Solution. Make a sketch! The crucial thing to realize that we are asked for **total area** and not **net area**. We therefore need to split up the interval $[-3, 2]$ into pieces according to where y is positive and negative. Since $f(x) = -x(x+2)$, we split $[-3, 2]$ into $[-3, -2]$ (where $f(x) < 0$), $[-2, 0]$ (where $f(x) > 0$), and $[0, 2]$ (where $f(x) < 0$).

$$\begin{aligned}
\text{total area } = \int_{-3}^{2} |f(x)| \mathrm{d}x &= \int_{-3}^{-2} |f(x)| \mathrm{d}x + \int_{-2}^{0} |f(x)| \mathrm{d}x + \int_{0}^{2} |f(x)| \mathrm{d}x \\
&= -\int_{-3}^{-2} f(x) \mathrm{d}x + \int_{-2}^{0} f(x) \mathrm{d}x - \int_{0}^{2} f(x) \mathrm{d}x \\
&= -\left[-\frac{1}{3}x^3 - x^2\right]_{-3}^{-2} + \left[-\frac{1}{3}x^3 - x^2\right]_{-2}^{0} - \left[-\frac{1}{3}x^3 - x^2\right]_{0}^{2} \\
&= \frac{4}{3} + \frac{4}{3} + \frac{20}{3} = \frac{28}{3}
\end{aligned}$$