Example 106. You and your dog Elvis are standing at the water's edge when you throw the ball 5 m deep into the sea and 10 m down the shore (so that it is at a distance of $\sqrt{5^2+10^2}$ meters). You know that Elvis runs at about 6 m/sec on land, and that his swimming speed is about 1 m/sec. Which path should Elvis take if his goal is to minimize the time to the ball?

Solution. Let x (in m) be the distance run on shore (takes time $x/6\text{ sec}$).

Then the distance swam in water is $w=\sqrt{5^2+(10-x)^2}$ (which takes time w).

Hence, the total time is $t=\frac{x}{6}+\sqrt{5^2+(10-x)^2}$. Our goal is to minimize t for x in $[0,10]$.

$$\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{1}{6}-\frac{2\cdot(10-x)}{2\sqrt{5^2+(10-x)^2}}=\frac{\sqrt{5^2+(10-x)^2}-6(10-x)}{6\sqrt{5^2+(10-x)^2}}$$

Thus, $\frac{\mathrm{d}t}{\mathrm{d}x}=0$ is equivalent to $5^2+(10-x)^2=36(10-x)^2$ or $5=7(10-x)^2$, so that $10-x=\pm\sqrt{\frac{5}{7}}$.

Of the critical points $x=10\pm\sqrt{\frac{5}{7}}$, only $x=10-\sqrt{\frac{5}{7}}$ is in $[0,10]$.

The endpoints $x=0$ and $x=10$ are clearly not minimizing t (indeed, for $x=0$, $t=\sqrt{125}\approx 11.180$, and, for $x=10$, $t=\frac{5}{3}+5\approx 6.667$), so that the absolute minimum of t must occur at the only critical point $x=10-\sqrt{\frac{5}{7}}\approx 9.155\text{ m}$ (for which $t\approx 6.597\text{ sec}$).

Comment. The calculation looks a bit lighter if we let x be the meters swum in the water...

See. **Do Dogs Know Calculus?** Timothy J. Pennings. `http://www.maa.org/features/elvisdog.pdf`

Example 107. (cf. Exercise 4.6.11) You are designing a rectangular poster to contain 50 in^2 of printing with a 4 in margin at the top and bottom, and a 2 in margin at each side. What overall dimensions will minimize the amount of paper used?

Solution. Suppose the printed part of the poster has height x (in) and width y (in).

Our goal is to minimize $A=(x+8)(y+4)$. The constraint is $xy=50$.

From the constraint, $y=\frac{50}{x}$, so we need to minimize $A=(x+8)\left(\frac{50}{x}+4\right)=82+4x+\frac{400}{x}$ (with x in $(0,\infty)$).

$\frac{\mathrm{d}A}{\mathrm{d}x}=4-\frac{400}{x^2}=0$ is equivalent to $x^2=100$, so that $x=\pm 10$, of which only $x=10$ is practical.

Since values of x close to 0 and ∞ (the endpoints) are clearly not minimizing A, the absolute minimum of A occurs at $x=10$ (the only critical point). The corresponding optimal overall dimensions for the poster are height $x+8=18\text{ in}$ and width $y+4=9\text{ in}$.

Solution. (alternative setup) Suppose the overall poster has height x (in) and width y (in).

Our goal is to minimize $A=x\cdot y$. The constraint is $(x-8)(y-4)=50$.

From the constraint, $y=\frac{50}{x-8}+4$, so we need to minimize $A=x\left(\frac{50}{x-8}+4\right)=4x+\frac{50x}{x-8}$ (with x in $(8,\infty)$).

$\frac{\mathrm{d}A}{\mathrm{d}x}=4+\frac{50(x-8)-50x}{(x-8)^2}=0$ is equivalent to $4(x-8)^2+50(x-8)-50x=4(x-18)(x+2)=0$, so that $x=18$ or $x=-2$, of which only $x=18$ is practical.

Since values of x close to 8 and ∞ (the endpoints) are clearly not minimizing A, the absolute minimum of A occurs at $x=18$ (the only critical point). The corresponding optimal overall dimensions for the poster are height $x=18\text{ in}$ and width $y=\frac{50}{x-8}+4=9\text{ in}$.