Example 104. (cf. Example 4.6.1) An open box is folded from a 6 in by 4 in piece of cardboard by cutting congruent squares from the corners and bending up the sides. How large should the cutout squares be to make the box hold as much as possible?

Solution. Let x be the side length of the squares cut from the corners. Then the volume of the box is

$$V = x(6-2x)(4-2x) = 4x^3 - 20x^2 + 24x.$$

Note that x can range from $x=0$ (zero volume) to $x=2$ (zero volume, again). We want to find the absolute maximum of V for x in $[0,2]$.

Since $\frac{\mathrm{d}V}{\mathrm{d}x} = 12x^2 - 40x + 24 = 4(3x^2 - 10x + 6)$, solving $\frac{\mathrm{d}V}{\mathrm{d}x} = 0$ yields $x = \frac{10 \pm \sqrt{10^2 - 4 \cdot 18}}{6} = \frac{5 \pm \sqrt{7}}{3}$ for the critical points. Since $\frac{5+\sqrt{7}}{3} \approx 2.549$, the only critical point in $[0,2]$ is $x = \frac{5-\sqrt{7}}{3} \approx 0.785$.

Since $V=0$ at the endpoints, the absolute max of V over $[0,2]$ must occur at a critical point, meaning the absolute max must be at $x = \frac{5-\sqrt{7}}{3}$.

In conclusion, for a maximum volume box, the cutout squares should have side length $\frac{5-\sqrt{7}}{3} \approx 0.785 \text{ in}$.

Comment. The maximum volume is $\frac{8}{27}(10 + 7\sqrt{7}) \approx 8.450 \text{ in}^3$. For comparison, 1 in cutout squares would result in a box with volume $1 \cdot 4 \cdot 2 = 8 \text{ in}^3$.

Example 105. A small rectangular garden of area 80 square meters is to be surrounded on three sides by a brick wall costing 5 dollars per meter and on one side by a fence costing 3 dollars per meter. Find the dimensions of the garden such that the overall cost is minimized.

Solution. Let a be the length in meters of the side with a fence, and b the length of the other side.

Then, the overall cost is $C = (5+3)a + (5+5)b = 8a + 10b$. (This is the objective function.)

On the other hand, we have $ab = 80$. (This is a constraint equation.)

In order to minimize the cost, we express cost as a function of a. Since $b = \frac{80}{a}$ (because $ab = 80$), we get that the cost is $C(a) = 8a + 10 \cdot \frac{80}{a} = 8a + 800a^{-1}$.

Our task is to find the absolute minimum of $C(a)$ for a in $(0, \infty)$.

$C'(a) = 8 + 800 \cdot (-a^{-2}) = 8 - 800a^{-2}$.

We now solve $C'(a) = 0$ to find the critical values: $8 - 800a^{-2} = 0$ simplifies to $a^2 = 100$, which implies $a = \pm 10$. Therefore, the only critical point in $(0, \infty)$ is $a = 10$.

After determining that there is a local min (and, hence, global min; that's because there are no other critical points in $(0, \infty)$) at $a = 10$, we conclude that, to minimize costs, the length of the side with a fence should be $a = 10$ meters and the length of the other side should be $b = \frac{80}{a} = 8$ meters.

To determine that there is indeed a local min at $a = 10$, we have several options:

(a) Observe that for small a (close to 0) and large a, the cost is definitely not optimal (actually the cost becomes arbitrarily large); hence, the absolute minimum must be somewhere in between, and the only candidate is $a = 10$.

(b) Apply the second-derivative test: $C''(a) = 1600a^{-3}$, so that $C''(10) = \frac{8}{5} > 0$, which shows that there is a local min at $a = 10$.

(c) Apply the first-derivative test: since, say, $C''(1) = -792 < 0$ and $C''(20) = 6 > 0$, we conclude that C' changes from $-$ to $+$ at $a = 10$, which again shows that there is a local min at $a = 10$.

Comment. We could also have expressed the cost as a function of b. Then $C(b) = 8 \cdot \frac{80}{b} + 10b = 640b^{-1} + 10b$ and $C'(b) = -640b^{-2} + 10$, so that $C'(b) = 0$ simplifies to $b^2 = 64$. We would conclude that $b = 8$ and then determine $a = \frac{80}{b} = 10$, ending up (of course!) with the same dimensions as before.

Armin Straub
straub@southalabama.edu