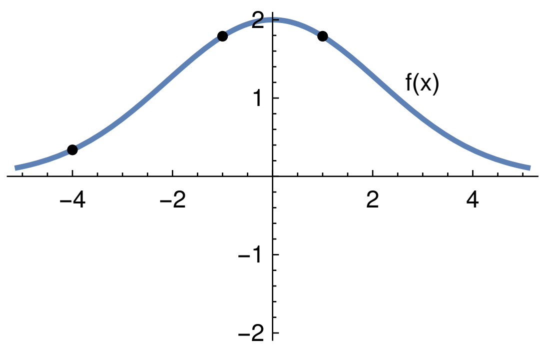
Review.

- f' tells us whether f is increasing or decreasing.
- f'' tells us whether f is concave up or concave down.
- An inflection point (where, by definition, concavity changes) is a local extremum of the derivative.
- first-derivative test vs second-derivative test

Example 96. Use the graph below to fill in each entry of the grid with positive, negative or zero.



	$f(x)$	$f'(x)$	$f''(x)$
$x=-4$	$+$	$+$	$+$
$x=1$	$+$	$+$	$-$
$x=1$	$+$	$-$	$-$

Example 97. Determine the local extrema of $f(x)=x^4$.

Solution. (graphical) From the shape of this familiar graph we know that there is a local minimum at $x=0$ (which is an absolute minimum).

Solution. (second-derivative test; inconclusive) Since $f'(x)=4x^3$, solving $f'(x)=0$, we find that $x=0$ is the only critical point. Using $f''(x)=12x^2$, we obtain $f''(0)=0$ which does not allow us to conclude whether $x=0$ is a minimum, maximum or neither.

Comment. Looking a little closer, we can see that $f''(x)>0$ for all $x\neq 0$. This shows that f is concave up everywhere, which, in particular, implies that $x=0$ must be a minimum.

Solution. (first-derivative test) Again, since $f'(x)=4x^3$, solving $f'(x)=0$, we find that $x=0$ is the only critical point. We choose two points, say $x=-1$ and $x=1$, around $x=0$ to see whether f' changes sign: since $f'(-1)=-4<0$ and $f'(1)=4>0$, we see that f' changes from $-$ to $+$ at $x=0$. This implies that there is a minimum at $x=0$.

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Example 98. (again, this time with focus on f'') Consider $f(x) = x e^x$.

(a) Using the second-derivative test, determine all local extrema of f.

(b) Determine the inflection points of f.

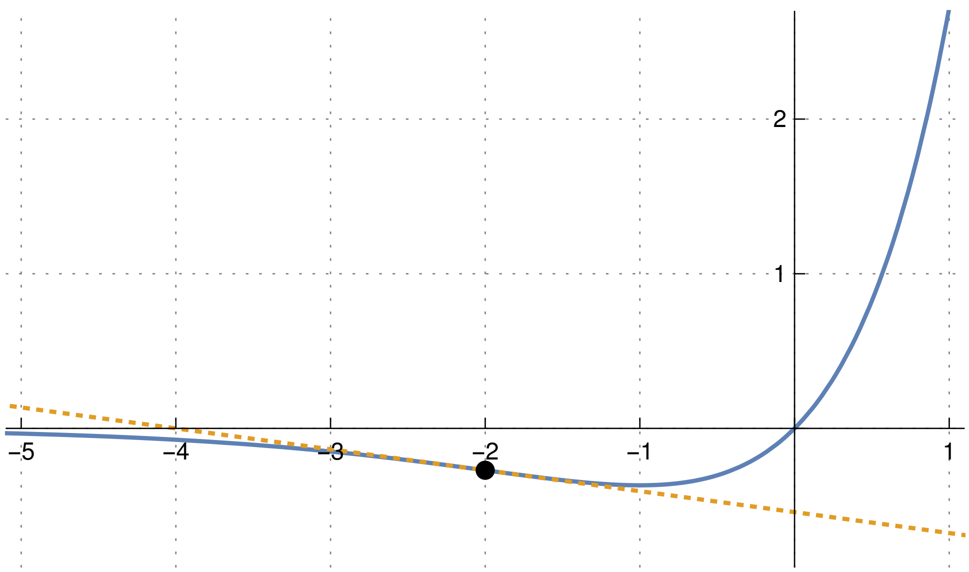
Note. We discussed the extrema of this function on $[-2, 2]$ in Example 88 (using the first-derivative test).

Solution.

(a) As earlier, we compute $f'(x) = 1 \cdot e^x + x \cdot e^x = (x+1)e^x$.
Solving $f'(x) = 0$, we find that the only critical point is at $x = -1$.
Since $f''(x) = (x+2)e^x$, we have $f''(-1) = e^{-1} > 0$ (concave up), which implies that $x = -1$ is a local minimum.

(b) Solving $f''(x) = 0$, we find that the only candidate for an inflection point is at $x = -2$.
We choose two points, say $x = -3$ and $x = -1$, around $x = 0$ to see whether f'' changes sign: since $f''(-3) = -e^{-3} < 0$ and $f'(-1) = e^{-1} > 0$, we conclude that there is an inflection point at $x = -2$.

The plot. Note how, at the inflection point at $x = -2$, the graph lies neither above (concave up) nor below (concave down) the tangent line.



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