Example 89. Find the extreme values of $f(x) = x^2 + 3x + 5$.

Solution. We know that this is a parabola (opening upwards) so that there is an absolute minimum and no further extrema. Since $f'(x) = 2x + 3$, solving $f'(x) = 0$, we find that the only critical point is at $x = -\frac{3}{2}$. Since this is the only candidate, the absolute minimum must be at $x = -\frac{3}{2}$.

The first-derivative test

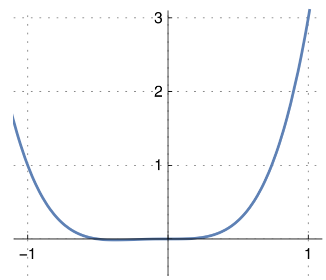
If $f' > 0$ (resp. $f' < 0$) on an interval, then f is **increasing** (resp. **decreasing**) on that interval.

A function is called monotonic on an interval if it is either increasing or decreasing on that interval.

(first-derivative test) Suppose $f'(a) = 0$ and that f is differentiable near $x = a$.

- If $f'(x)$ changes from > 0 to < 0 at $x = a$, then $f(x)$ has a local maximum at $x = a$.
- If $f'(x)$ changes from < 0 to > 0 at $x = a$, then $f(x)$ has a local minimum at $x = a$.
- If $f'(x)$ does not change sign at $x = a$, then $f(x)$ has no local extremum at $x = a$.

Example 90. Consider $f(x) = x^3 + 2x^4$.



(a) Find all critical points of $f(x)$.

(b) Then find all extrema (both local and absolute) when f is restricted to the interval $[-1, 1]$.

(c) On which intervals is $f(x)$ increasing/decreasing?

Comment. This function is plotted to the right. That plot, however, is possibly misleading (at a glance it might seem like there is minimum at $x = 0$). After crunching the numbers, compare with the second plot below.

Solution.

(a) $f'(x) = 3x^2 + 8x^3 = x^2(3 + 8x)$
Solving $f'(x) = 0$, we find that the critical points are 0 and $-\frac{3}{8}$.

(b) The extreme values can only occur at 0, $-\frac{3}{8}$ (critical points) or at -1, 1 (endpoints).

x	-1		$-\frac{3}{8}$		0		1
$f(x)$	1	$\searrow$	$-\frac{27}{2048} \approx -0.013$	$\nearrow$	0	$\nearrow$	3
$f'(x)$		$-$	0	$+$	0	$+$	

Hence, f on $[-1, 1]$ has an absolute maximum of 3 at $x = 1$ and an absolute minimum of $-\frac{27}{2048}$ at $x = -\frac{3}{8}$. It further has a local maximum of 1 at $x = -1$.

(c) f is increasing on the open interval $(-3/8, \infty)$ and decreasing on the open interval $(-\infty, -3/8)$.

A plot that confirms our findings.

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Example 91. Consider $f(x) = \frac{x+1}{x^2+x+4}$.

(a) Find all critical points of $f(x)$.

(b) Then find all (local and absolute) extrema when f is restricted to the interval $[-4, 4]$.

(c) On which intervals is $f(x)$ increasing/decreasing?

Note. The denominator is never zero (for real x) so that the natural domain of f is $(-\infty, \infty)$.

Solution.

(a) $f'(x) = \frac{-(x-1)(x+3)}{(x^2+x+4)^2}$

Solving $f'(x) = 0$, we find that the critical points are 1 and -3.

(b) The extreme values can only occur at 1, -3 (critical points) or at -4, 4 (endpoints).

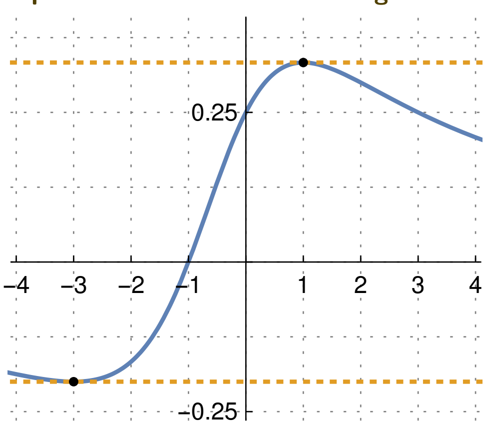
x	-4		-3		1		4
$f(x)$	$-\frac{3}{16} = -0.1875$	$\searrow$	$-\frac{1}{5}$	$\nearrow$	$\frac{1}{3}$	$\searrow$	$\frac{5}{24} \approx 0.2083$
$f'(x)$		$-$	0	$+$	0	$-$	

In conclusion:

- There is an absolute minimum of $-\frac{1}{5}$ at $x = -3$ and an absolute maximum of $\frac{1}{3}$ at $x = 1$.
- There is an additional local minimum of $\frac{5}{24}$ at $x = 4$ and a local maximum of $-\frac{3}{16}$ at $x = -4$.

(c) f is increasing on the open interval $(-3, 1)$ and decreasing on the open intervals $(-\infty, -3)$ and $(1, \infty)$.

A plot that confirms our findings.



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