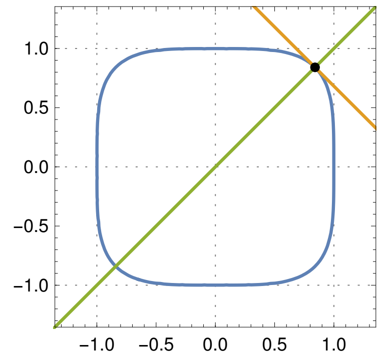
Example 71. Consider the curve $x^4+y^4=1$.

(a) Determine $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(b) Determine the lines tangent and normal to the curve at the point $\left(\frac{1}{\sqrt[4]{2}},\frac{1}{\sqrt[4]{2}}\right)$.



Comment. This curve is called a **squircle** (dinner plates, phone buttons, applied in optics, ...).
https://en.wikipedia.org/wiki/Squircle

Solution.

(a) Applying $\frac{d}{dx}$ to both sides of $x^4+y^4=1$, we obtain $4x^3+4y^3\frac{dy}{dx}=0$, so that $\frac{dy}{dx}=-\frac{x^3}{y^3}$. Consequently, by the quotient rule,

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left[-\frac{x^3}{y^3}\right]=-\frac{3x^2\cdot y^3-x^3\cdot 3y^2\frac{dy}{dx}}{(y^3)^2}=-\frac{3x^2y^3+3x^6y^{-1}}{y^6}=-\frac{3x^2}{y^7}(y^4+x^4)=-\frac{3x^2}{y^7}.$$

In the final step, we simplified using $x^4+y^4=1$.

(b) The slope of the line tangent to the curve at that point is $\left[\frac{dy}{dx}\right]_{x=2^{-1/4},y=2^{-1/4}}=-1$.

Hence, the tangent line has equation $\left(y-\frac{1}{\sqrt[4]{2}}\right)=-\left(x-\frac{1}{\sqrt[4]{2}}\right)$, or, $y=-x+2^{3/4}$.

The normal line has equation $\left(y-\frac{1}{\sqrt[4]{2}}\right)=+\left(x-\frac{1}{\sqrt[4]{2}}\right)$, or, $y=x$.

Why was this (geometrically) clear from the beginning?! See the sketch above.

Derivatives of inverse functions

- $\frac{d}{dx}f^{-1}(x)=\frac{1}{f'(f^{-1}(x))}$ **(derivative of inverse functions)**

Why? Differentiating both sides of $f(f^{-1}(x))=x$ and using the chain rule, we find that

$$\frac{d}{dx}f(f^{-1}(x))=f'(f^{-1}(x))\frac{d}{dx}f^{-1}(x)=1.$$

Comment. Rather than memorizing a formula for $\frac{d}{dx}f^{-1}(x)$, it is advisable to remember to apply the chain rule to the defining equation (see examples).
[You can also think of it as using implicit differentiation on $f(y)=x$ (instead of $y=f^{-1}(x)$) to find $\frac{dy}{dx}$.]

Example 72. (derivative of $\ln(x)$) It follows from $e^{\ln(x)}=x$ that

$$\frac{d}{dx}e^{\ln(x)}=e^{\ln(x)}\frac{d}{dx}\ln(x)=1 \quad\Longrightarrow\quad \frac{d}{dx}\ln(x)=\frac{1}{e^{\ln(x)}}=\frac{1}{x}.$$

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- $\frac{\mathrm{d}}{\mathrm{d}x}\ln(x) = \frac{1}{x}$ **(derivative of $\ln(x)$)**

Derivatives of other logarithms. To find the derivative of $\log_a(x)$, recall that $\log_a(x) = \frac{\ln(x)}{\ln(a)}$. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}\log_a(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\ln(x)}{\ln(a)} = \frac{1}{x\ln(a)}.$$

Alternatively. Use $\frac{\mathrm{d}}{\mathrm{d}x}a^x = \ln(a)\, a^x$ to derive the derivative of $\log_a(x)$, the inverse function of a^x.

(logarithmic differentiation) $\frac{\mathrm{d}y}{\mathrm{d}x} = y\frac{\mathrm{d}}{\mathrm{d}x}\ln(y)$

Why? This is equivalent to $\frac{\mathrm{d}}{\mathrm{d}x}\ln(y) = \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}$.

Comment. This is another formula not to memorize. Rather, when taking the derivative of a product y (or other function that simplifies when the logarithm is applied), just remember to differentiate $\ln(y)$ instead.

Power rule for general exponents.

$x^n = e^{\ln(x^n)} = e^{n\ln(x)}$ allows us to **define** x^n for any real exponent n (and $x > 0$).

We then find $\frac{\mathrm{d}}{\mathrm{d}x}x^n = \frac{\mathrm{d}}{\mathrm{d}x}e^{n\ln(x)} = e^{n\ln(x)}\frac{\mathrm{d}}{\mathrm{d}x}[n\ln(x)] = x^n \cdot \frac{n}{x} = nx^{n-1}$, the familiar power rule.

Example 73. Determine $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt[3]{\frac{(x-1)(x^2+2)}{x+2}}$.

Note. We could compute this derivative using the chain rule combined with the product and quotient rule. However, this is quite a bit of work (do it for practice!). Logarithmic differentiation is much quicker and provides a cleaner answer. The reason logarithmic differentiation is beneficial here is that our function breaks into simpler terms when applying the logarithm (see solution).

Solution. (using logarithmic differentiation) Let $y = \sqrt[3]{\frac{(x-1)(x^2+2)}{x+2}}$.

Then $\ln(y) = \frac{1}{3}[\ln(x-1) + \ln(x^2+2) - \ln(x+2)]$. Differentiating both sides, we obtain

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3}\left[\frac{1}{x-1} + \frac{2x}{x^2+2} - \frac{1}{x+2}\right] \quad \Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3}\left[\frac{1}{x-1} + \frac{2x}{x^2+2} - \frac{1}{x+2}\right]\sqrt[3]{\frac{(x-1)(x^2+2)}{x+2}}.$$