Example 66. Determine $\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{g(x)}$ (in terms of $g(x)$ and $g'(x)$).

Solution. (using quotient rule) We apply $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$ with $f(x) = 1$ and $f'(x) = 0$:
$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{g(x)} = \frac{0 \cdot g(x) - 1 \cdot g'(x)}{g(x)^2} = -\frac{g'(x)}{g(x)^2}$

Solution. (using chain rule) We apply $\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = f'(g(x))g'(x)$ with $f(x) = \frac{1}{x}$ and $f'(x) = -\frac{1}{x^2}$:
$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{g(x)} = -\frac{1}{g(x)^2} \cdot g'(x) = -\frac{g'(x)}{g(x)^2}$

Comment. As illustrated by this example, the quotient rule can be deduced from the product rule and the chain rule.

Important. In the sequel, it will be convenient to write just y instead of $g(x)$.
Instead of $\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{g(x)} = -\frac{g'(x)}{g(x)^2}$, for instance, we would write $\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{y} = -\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x}$.

Example 67. Determine $\frac{\mathrm{d}}{\mathrm{d}x} y^3$.

Solution. By the chain rule, $\frac{\mathrm{d}}{\mathrm{d}x} y^3 = 3y^2 \frac{\mathrm{d}y}{\mathrm{d}x}$.

Important. $\frac{\mathrm{d}}{\mathrm{d}y} y^3 = 3y^2$ and the chain rule allows us to write $\frac{\mathrm{d}}{\mathrm{d}x}\ldots = \left[\frac{\mathrm{d}}{\mathrm{d}y}\ldots\right]\frac{\mathrm{d}y}{\mathrm{d}x}$.

(normal lines) The line **normal** to a curve at a point P, is the line through P which is perpendicular to the tangent line.

Review. Two lines are perpendicular if the product of their slopes is -1.
Hence, if the tangent line has slope m, then the normal line has slope $-\frac{1}{m}$.

Implicit differentiation

Example 68. Consider the unit circle $x^2 + y^2 = 1$. Determine $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Review. The equation for a circle of radius r and center (a, b) is $(x - a)^2 + (y - b)^2 = r^2$.

Solution. (without implicit differentiation) The circle is composed of the graphs of the two functions $y = \sqrt{1 - x^2}$ (upper half-circle) and $y = -\sqrt{1 - x^2}$ (lower half-circle).
Hence, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\pm\sqrt{1 - x^2}\right] = \pm\frac{1}{2\sqrt{1 - x^2}}\frac{\mathrm{d}}{\mathrm{d}x}[1 - x^2] = \pm\frac{-x}{\sqrt{1 - x^2}}$.
[These are two formulas in one, but we can combine them by noticing that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-x}{\pm\sqrt{1 - x^2}} = -\frac{x}{y}$.]

Solution. (with implicit differentiation) Applying $\frac{\mathrm{d}}{\mathrm{d}x}$ to both sides of $x^2 + y^2 = 1$, we obtain $2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, so that $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$.

Important. We are using $\frac{\mathrm{d}}{\mathrm{d}x} y^2 = 2y \cdot \frac{\mathrm{d}y}{\mathrm{d}x}$ (see examples at beginning of class). That's the chain rule!

Solution. (geometrically) Make a sketch! It is clear that the normal line at (x, y) [a point on the circle] has slope $\frac{y}{x}$. It follows that the tangent line must have slope $-\frac{x}{y}$, so that $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$.

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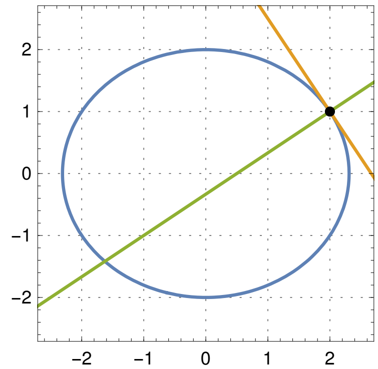

Example 69. Consider the curve $3x^2 + 4y^2 = 16$.

(a) Determine $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(b) Determine the lines tangent and normal to the curve at the point $(2, 1)$.

Solution.

(a) Applying $\frac{d}{dx}$ to both sides of $3x^2 + 4y^2 = 16$, we obtain $6x + 8y\frac{dy}{dx} = 0$, so that $\frac{dy}{dx} = -\frac{3x}{4y}$.
Consequently, by the quotient rule,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[-\frac{3}{4}\frac{x}{y}\right] = -\frac{3}{4}\frac{y - x\frac{dy}{dx}}{y^2} = -\frac{3}{4}\frac{y - x\left(-\frac{3}{4}\frac{x}{y}\right)}{y^2} = -\frac{3}{16}\frac{4y^2 + 3x^2}{y^3} = -\frac{3}{y^3}.$$

In the final step, we simplified using $3x^2 + 4y^2 = 16$.

(b) The slope of the line tangent to the curve at $(2, 1)$ is $\left[\frac{dy}{dx}\right]_{x=2, y=1} = \left[-\frac{3x}{4y}\right]_{x=2, y=1} = -\frac{6}{4} = -\frac{3}{2}$.
Hence, the tangent line has equation $(y - 1) = -\frac{3}{2}(x - 2)$, which simplifies to $y = -\frac{3}{2}x + 4$.
The normal line has slope $\frac{2}{3}$ and, thus, equation $(y - 1) = \frac{2}{3}(x - 2)$, or, $y = \frac{2}{3}x - \frac{1}{3}$.

Comment. This curve is an ellipse.

Example 70. Consider the curve $e^{3x} = \sin(x^2 + 5y^2)$. Determine $\frac{dy}{dx}$.

Solution. Applying $\frac{d}{dx}$ to both sides of $e^{3x} = \sin(x^2 + 5y^2)$, we obtain

$$3e^{3x} = \cos(x^2 + 5y^2)\frac{d}{dx}[x^2 + 5y^2] = \cos(x^2 + 5y^2)\left(2x + 10y\frac{dy}{dx}\right),$$

so that $2x + 10y\frac{dy}{dx} = \frac{3e^{3x}}{\cos(x^2 + 5y^2)}$ and, therefore, $\frac{dy}{dx} = \frac{3e^{3x}}{10y\cos(x^2 + 5y^2)} - \frac{x}{5y}$.

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