Example 49.

(a) Compute $f'(x)$ for $f(x) = \frac{1}{x^2}$.

(b) Determine the line tangent to the graph of $f(x)$ at $x = 1$.

Solution.

(a) We need to determine $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ for $f(x) = \frac{1}{x^2}$.
Note that

$$f(x+h) - f(x) = \frac{1}{(x+h)^2} - \frac{1}{x^2} = \frac{x^2 - (x+h)^2}{(x+h)^2 x^2} = \frac{-2hx - h^2}{(x+h)^2 x^2},$$

so that

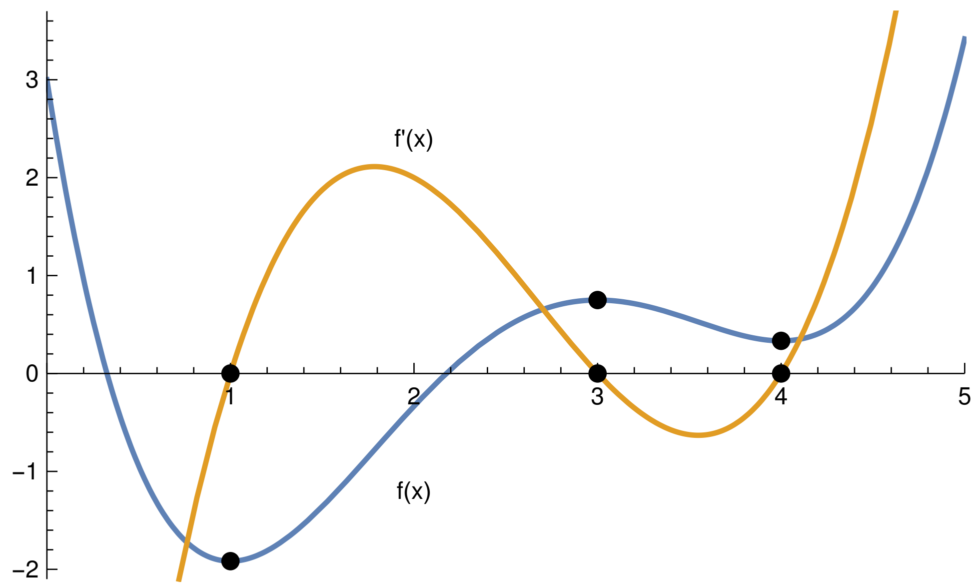
$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{-2hx - h^2}{(x+h)^2 x^2 h} = \lim_{h \to 0} \frac{-2x - h}{(x+h)^2 x^2} = \frac{-2x - 0}{(x+0)^2 x^2} = -\frac{2x}{x^4} = -\frac{2}{x^3}.$$

Important. As we did in class, sketch both $f(x)$ and $f'(x)$ and make sure you see the relation between the two.

(b) From the first part, the slope of that line is $f'(1) = -2$. It also passes through $(1, f(1)) = (1, 1)$. Hence, it has the equation $y - 1 = -2(x - 1)$, which simplifies to $y = -2x + 3$.

Example 50. Using some tool, plot $f(x) = \frac{1}{4}x^4 - \frac{8}{3}x^3 + \frac{19}{2}x^2 - 12x + 3$. Then sketch $f'(x)$.

Solution. $f(x)$ is plotted in blue. Make sure that you can (roughly) sketch $f'(x)$ from that by hand!



Important comment. Notice how we have $f'(x) = 0$ precisely at the (local) minima/maxima of $f(x)$. This crucial observation will allow us to find these points of special interest.

Example 51. If $f(t)$ describes the temperature in $°\mathrm{F}$ at time t in h since 6AM this morning. What is measured by $f'(t)$ and what are the units? Interpret the value $f'(2) = 4$.

Solution. $f'(t)$ describes the **rate of change** of the temperature over time. The units of $f'(t)$ are $\frac{\mathrm{F}}{\mathrm{h}}$.
$f'(2) = 4$ means that, at 8AM, the temperature is increasing at a rate of $4\ \mathrm{F/h}$ (meaning that, if that rate didn't change, it will be $4\ \mathrm{F}$ warmer at 9AM).

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Theorem 52. If $f(x)$ is differentiable at $x=x_0$, then $f(x)$ is continuous at $x=x_0$.

This implies that a function with a discontinuity at $x=x_0$ is not differentiable at x_0.
Other typical reasons for a function to not be differentiable are *corners/cusps* and *vertical tangents* (see the next two examples for instances of each).

Proof of theorem. Suppose f is differentiable at x. Since $f(x+h)=f(x)+h\frac{f(x+h)-f(x)}{h}$, it follows that

$$\lim_{h\to 0} f(x+h) = \lim_{h\to 0}\left[f(x)+h\frac{f(x+h)-f(x)}{h}\right] = f(x)+0\cdot f'(x) = f(x).$$

This shows (why?!!) that f is continuous at x.
[This is a precise way of observing that $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ (differentiability at x) can only exist if $\lim_{h\to 0}[f(x+h)-f(x)]=0$ (continuity at x).]

Example 53. (corner) Make a sketch of $f(x)=|x|$ and observe that it has a corner at $x=0$.

- $f(x)$ is differentiable for all $x\neq 0$.
- $f(x)$ is continuous for all x.

See final example for

Example 54. (vertical tangent) Make a sketch of $f(x)=\sqrt[3]{x}$ and observe that it has a vertical tangent line at $x=0$.

- $f(x)$ is differentiable for all $x\neq 0$.
- $f(x)$ is continuous for all x.

Example 55. Compute $f'(x)$ for $f(x)=|x|$ for all x where $f(x)$ is differentiable.

Solution. Note that $f(x)=\begin{cases} x, & x\geqslant 0,\\ -x, & x<0.\end{cases}$

- If $x>0$, then $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)-x}{h}=\lim_{h\to 0}\frac{h}{h}=1.$
- If $x<0$, then $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{-(x+h)-(-x)}{h}=\lim_{h\to 0}\frac{-h}{h}=-1.$
- If $x=0$, then
  - $\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{(0+h)-0}{h}=\lim_{h\to 0^+}\frac{h}{h}=1,$
  - $\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{-(0+h)-(-0)}{h}=\lim_{h\to 0^+}\frac{-h}{h}=-1,$

  so that $\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$ does not exist. Hence, $f(x)$ is not differentiable at $x=0$.

In conclusion, we have $f'(x)=\begin{cases} 1, & x>0,\\ -1, & x<0.\end{cases}$

Comment. Note that $f(x)$ is piecewise a line, so that $f'(x)$ will be just the slopes (1 if $x>0$, and -1 if $x<0$) of those two lines.

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