- $f(x)$ has $y=a$ as a **horizontal asymptote** if $\lim_{x\to\infty} f(x)=a$ or if $\lim_{x\to-\infty} f(x)=a$.
- $f(x)$ has $x=a$ as a **vertical asymptote** if $\lim_{x\to a^-} f(x)=\pm\infty$ or if $\lim_{x\to a^+} f(x)=\pm\infty$.

Example 39. Determine $\lim_{x\to\infty} \frac{2\sqrt{x}-3x^2+1}{5x^2+4}$.

Solution. $\lim_{x\to\infty} \frac{2\sqrt{x}-3x^2+1}{5x^2+4} = -\frac{3}{5}$ (fill in the details!)

Comment. Hence, $f(x)=\frac{2\sqrt{x}-3x^2+1}{5x^2+4}$ has the line $y=-\frac{3}{5}$ as a **horizontal asymptote**.

Example 40. Determine $\lim_{x\to\infty} \sqrt[3]{\frac{2\sqrt{x}-3x^2+1}{5x^2+4}}$.

Solution. $\lim_{x\to\infty} \sqrt[3]{\frac{2\sqrt{x}-3x^2+1}{5x^2+4}} = \sqrt[3]{-\frac{3}{5}} = -\sqrt[3]{\frac{3}{5}}$

Here, we used that $\lim_{x\to c} \sqrt[3]{f(x)} = \sqrt[3]{L}$ with $L=\lim_{x\to c} f(x)$, which holds because $\sqrt[3]{x}$ is continuous at $x=L$ (in fact, it is continuous everywhere).

Example 41. Determine $\lim \frac{x^2+2x-3}{x^2-1}$ as $x\to 1^{\pm}$ and as $x\to -1^{\pm}$.

Solution. First, let us note that $\frac{x^2+2x-3}{x^2-1} = \frac{(x-1)(x+3)}{(x-1)(x+1)} = \frac{x+3}{x+1}$.

Hence, $\lim_{x\to 1} \frac{x^2+2x-3}{x^2-1} = \lim_{x\to 1} \frac{x+3}{x+1} = 2$.

On the other hand, $\lim_{x\to -1^+} \frac{x^2+2x-3}{x^2-1} = \lim_{x\to -1^+} \frac{x+3}{x+1} = \frac{2}{0^+} = +\infty$.

Likewise, $\lim_{x\to -1^-} \frac{x^2+2x-3}{x^2-1} = \lim_{x\to -1^-} \frac{x+3}{x+1} = \frac{2}{0^-} = -\infty$.

Comment. Hence, $f(x)=\frac{x^2+2x-3}{x^2-1}$ has a **vertical asymptote** at $x=-1$.

Example 42. Determine $\lim_{x\to\infty} \left(2x^3 - x\sqrt{4x^4+3x}\right)$.

Solution. Note that $2x^3 - x\sqrt{4x^4+3x} = \frac{\left(2x^3 - x\sqrt{4x^4+3x}\right)\left(2x^3 + x\sqrt{4x^4+3x}\right)}{2x^3 + x\sqrt{4x^4+3x}} = \frac{-3x^3}{2x^3 + x\sqrt{4x^4+3x}}$.

Dividing by x^3, the biggest power of x (the powers are x^3, $x\sqrt{x^4}=x^3$ and $x\sqrt{x}=x^{3/2}$) in the denominator:

$$\frac{-3x^3}{2x^3 + x\sqrt{4x^4+3x}} = \frac{-3}{2+\sqrt{4+\frac{3}{x^3}}} \overset{x\to\infty}{\longrightarrow} \frac{-3}{2+\sqrt{4+0}} = -\frac{3}{4}$$

Hence, $\lim_{x\to\infty} \left(2x^3 - x\sqrt{4x^4+3x}\right) = -\frac{3}{4}$.