Review: our zoo of functions

- Polynomials:
  - linear functions: $f(x) = 2x - 3$
  - quadratic functions: $f(x) = x^2 + 2x - 3 = (x-1)(x+3)$
    [Review factoring and solving quadratic equations.]
  - cubic functions: $f(x) = 2x^3 - 3x^2 + x + 1$
  - higher degree: $f(x) = x^{17} - x^5 + 1$ (this is a degree 17 polynomial)
- Rational functions:

  These are just quotients of polynomials.
  - $f(x) = \frac{x+1}{x-1}$
  - $f(x) = \frac{x^2+3x+1}{x^3+1}$
- Algebraic functions:
  - square root: $f(x) = \sqrt{x} = x^{1/2}$

    For experts. Algebraic functions are those that appear when solving polynomial equations. For instance, $x^2 - 5 = 0$ has the solution $x = \sqrt{5}$.
- Exponential functions:
  - $f(x) = 2^x$ (2 is called the **base**)
  - $f(x) = e^x$ (with the natural base $e \approx 2.718$)

    We'll learn what makes this one of the most important functions.

  Important comment. We will mostly work with e^x and, for instance, replace 2^x with $2^x = e^{\ln(2) \cdot x}$.
- Trigonometric functions: (we will not discuss these)
  - $f(x) = \cos(x)$, $f(x) = \tan(x)$, ...

  For experts. These can actually be expressed using the exponential function!! (More in Calculus 2!) For instance, $\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$ where $i = \sqrt{-1}$ is the imaginary unit.
- Inverse functions:
  - logarithms: $f(x) = \log_2(x)$, $f(x) = \ln(x)$, ...

    Important comment. Again, we will mostly work with $\ln(x)$ and, for instance, replace $\log_2(x)$ with $\log_2(x) = \frac{\ln(x)}{\ln(2)}$.
  - inverse trig functions: $\sin^{-1}(x)$, $\tan^{-1}(x)$

    These are also commonly written as, for instance, $\arcsin(x) := \sin^{-1}(x)$.

Comment. Our book has the subtitle "Early Transcendentals", which refers to the fact that, from the beginning, we include transcendental functions (a technical term for nonalgebraic functions) like e^x, $\ln(x)$, $\cos(x)$, ...

Review: exponential functions and logarithms

Example 1. Sketch the graphs of 2^x, $\left(\frac{1}{2}\right)^x$ and e^x. How are they related?

Solution. Do it!

Since $\left(\frac{1}{2}\right)^x = 2^{-x}$, its graph is obtained from the graph of 2^x by reflecting through the y-axis.

Since $2^x = e^{\ln(2)x}$, its graph is a horizontal stretch of the graph of e^x. Since $e > 2$ (or, equivalently, $\ln(2) < 1$), 2^x increases slower than e^x.

Example 2. $b^8(2b)^4 = b^8 2^4 b^4 = 16 b^{12}$

Example 3. $\frac{(3y^3)^4}{2y^5} = \frac{81}{2}\frac{y^{12}}{y^5} = \frac{81}{2}y^7$

(Rules of exponents) Let $a > 0$ and $b > 0$. Then, for any real x, y,

$$a^x \cdot a^y = a^{x+y}, \quad \frac{a^x}{a^y} = a^{x-y}, \quad (a^x)^y = a^{xy}, \quad a^x \cdot b^x = (ab)^x, \quad \frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x.$$

Logarithms

If $f(x) = a^x$, then its **inverse function** is $f^{-1}(x) = \log_a x$.

Note. How do we know that $f(x)$ has an inverse function? Plot and check that it is 1-1!

By the very definition of logarithms, we therefore have

$$a^{\log_a x} = x \quad \text{and} \quad \log_a(a^x) = x.$$

(Rules of logarithms) Let $a > 0$. Then, for any positive real r, s,

- $\log_a(rs) = \log_a(r) + \log_a(s)$,
- $\log_a\left(\frac{r}{s}\right) = \log_a(r) - \log_a(s)$,
- $\log_a(r^t) = t \cdot \log_a(r)$.

Why? These rules follow from the rules for exponents. For instance, let's translate $a^{x+y} = a^x \cdot a^y$ into logarithmic form:

$a^x = r$ translates to $\log_a r = x$, and $a^y = s$ translates to $\log_a s = y$.

Hence, $a^{x+y} = rs$ translates to $\log_a(rs) = x + y$, that is $\log_a(rs) = \log_a(r) + \log_a(s)$.

How we can work with e^x and $\ln(x)$ instead of other bases

We will actually, almost exclusively, use the natural base $a = e$ (because, as we will seem it is indeed most natural for calculus).

Every other exponential function can be expressed using the natural base:

$$a^x = e^{\ln(a)x}$$

Why? Because $a = e^{\ln(a)}$ so that $a^x = (e^{\ln(a)})^x = e^{\ln(a)x}$.

Similarly for logarithms:

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}$$

Why? Using $x = a^{\log_a(x)}$, we find $\ln(x) = \ln(a^{\log_a(x)}) = \log_a(x)\ln(a)$. Now, divide by $\ln(a)$.

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Application: radioactive decay

Example 4. Carbon-14 has a half-life of 5750 years. Old bones were found to have lost 60% of their carbon-14. How old are they?

Solution. Let $y(t)$ be the amount [say, in grams] of carbon-14 in those bones after t years, with $y_0 = y(0)$ being the initial amount. Since radioactive decay is exponential,

$$y(t) = y_0\, e^{-rt},$$

where r is the decay rate. We can determine r because we know that $y(5750) = \frac{1}{2}y_0$:

$$y_0\, e^{-5750r} = \frac{1}{2}y_0 \quad \Longrightarrow \quad e^{-5750r} = \frac{1}{2} \quad \Longrightarrow \quad -5750r = \ln\left(\frac{1}{2}\right) \quad \Longrightarrow \quad r = \frac{\ln(2)}{5750}$$

Hence,

$$y(t) = y_0\, e^{-\frac{\ln(2)}{5750}t} = y_0 \cdot 2^{-t/5750} = y_0\left(\frac{1}{2}\right)^{t/5750}.$$

[Make sure that you understand how to convert between these equivalent expressions! For similar HW, the online system should expect either of these three.]

To determine the age of the bones, we solve $y(t) = \frac{4}{10} \cdot y_0$ for t:

$$y_0 \cdot 2^{-t/5750} = \frac{4}{10} \cdot y_0 \quad \Longrightarrow \quad 2^{-t/5750} = \frac{4}{10} \quad \Longrightarrow \quad -\frac{t}{5750} = \log_2\left(\frac{4}{10}\right) \quad \Longrightarrow \quad t \approx 7601.1$$

Comment. In hindsight, can you see why we could have immediately jumped to $y(t) = y_0\left(\frac{1}{2}\right)^{t/5750}$?

Comment. This method is known as **carbon dating**, developed in the 1940s by Willard Libby, who received the Nobel Prize in Chemistry in 1960.

Example 5. (lab!) Determine the domain and range of $f(x) = \frac{5}{1 - e^{3x}}$.

Review. $f(x)$ is a composition of two simpler functions! Namely, $f(x) = g(h(x))$ where $g(z) = \frac{5}{1 - z}$ and $h(x) = e^{3x}$. Recall that we also write $f = g \circ h$.

Solution. The domain of $f(x)$ consists of all x such that $1 - e^{3x} \neq 0$. Since $1 - e^{3x} = 0$ has only the solution $x = 0$, the domain of $f(x)$ is $\{x : x \neq 0\}$ or, in interval notation, $(-\infty, 0) \cup (0, \infty)$.

Since e^{3x} has range $(0, \infty)$, the range of $f(x)$ is the same as the range of $\frac{5}{1 - z}$ with z restricted to $(0, \infty)$.

Make a plot of $\frac{5}{1 - z}$! (How can it be obtained from the familiar plot of $\frac{1}{z}$?)

From that plot, we conclude that the range of $f(x)$ is $(-\infty, 0) \cup (5, \infty)$.

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