Diagonal and constant term representations of sequences

AN25—The Third Joint SIAM/CAIMS Annual Meetings

Minisymposium on Hypergeometric Series and Their Applications

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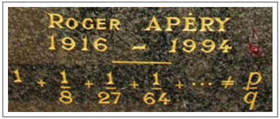
University of South Alabama

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \operatorname{diag} \frac{1}{(1-x-y)(1-z-w)-xyzw}$$

$$= \operatorname{ct} \left[\left(\frac{(x+y)(z+1)(x+y+z)(y+x+1)}{xyz} \right)^n \right]$$



based on joint work with:

Alin Bostan
(Université Paris-Saclay)

Sergey Yurkevich
(University of Vienna)

A simple example

EG **constant term**

$$\binom{2n}{n} = [x^n]\,(1+x)^{2n}$$

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EG constant term

$$\binom{2n}{n} = [x^n]\,(1+x)^{2n} = \operatorname{ct}[\boldsymbol{P}^n], \qquad \boldsymbol{P}(x) = \frac{(1+x)^2}{x}.$$

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EG diagonal

$\binom{2n}{n}$ is the diagonal of $\dfrac{1}{1-x-y}$

$$\sum_{n_1,\ldots,n_d \geqslant 0} a(n_1,\ldots,n_d)\, x_1^{n_1} \cdots x_d^{n_d} \qquad\qquad a(n,\ldots,n)$$

multivariate series — diagonal

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$\binom{2n}{n}$ is the diagonal of $\dfrac{1}{1-x-y}$

$$= \sum_{k=0}^{\infty} (x+y)^k$$

$$= \sum_{n,m\geqslant 0} \binom{m+n}{m} x^m y^n.$$

$$\sum_{n_1,\ldots,n_d\geqslant 0} a(n_1,\ldots,n_d)\, x_1^{n_1}\cdots x_d^{n_d} \qquad a(n,\ldots,n)$$

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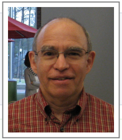
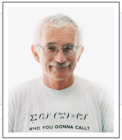
multivariate series — diagonal

THM Gessel, Zeilberger, Lipshitz 1981–88

Diagonals of rational functions are P-recursive.

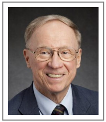



HW Constant terms are always diagonals.

Ramanujan's elliptic functions

- Berndt, Bhargava & Garvan (1995) develop Ramanujan's theories of elliptic functions based on the hypergeometric functions

$$ {}_2F_1\left(\frac{1}{m}, 1-\frac{1}{m}; 1; x\right), \qquad m \in \{2,3,4,6\}. $$



($m = 2$: classical; $m = 3, 4, 6$: alternative bases)

LEM Bostan, S, Yurkevich '23

Let $A_m(n) = \dfrac{\left(\frac{1}{m}\right)_n \left(1-\frac{1}{m}\right)_n}{n!^2}$ where $m \geqslant 2$ is an integer.

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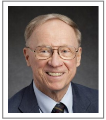

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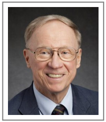

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2. $A_m(n)$ is a **constant term** if and only if $m \in \{2,3,4,6\}$.

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2. $A_m(n)$ is a **constant term** if and only if $m \in \{2,3,4,6\}$.

EG $m=3$

$$ 3^{3n} A_3(n) = \frac{(3n)!}{n!^3} = \binom{2n}{n}\binom{3n}{n} = \operatorname{ct}\left[\left(\frac{(1+x)^2(1+y)^3}{xy}\right)^n\right] $$

EG $m=5$

$5^{3n} A_5(n) = 1, 20, 1350, 115500, 10972500, \ldots$ is an integer sequence and diagonal but not a constant term.

Homework

- Such classifications are generally not straightforward!

EG **open!** Is the following hypergeometric sequence a constant term?

$$A(n) = \frac{(8n)!n!}{(4n)!(3n)!(2n)!} = \binom{8n}{4n}\binom{4n}{n}\binom{2n}{n}^{-1}$$

$$A(n) = 1, 140, 60060, 29745716, 15628090140, \ldots = \operatorname{ct}\left[\left(\frac{(1+x)^8}{(1-x)^2x^3}\right)^n\right]$$

not a Laurent polynomial so doesn't count as **constant term** today

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EG **open!** Is the following hypergeometric sequence a diagonal?

$$A(n) = \frac{\left(\frac{1}{9}\right)_n\left(\frac{4}{9}\right)_n\left(\frac{5}{9}\right)_n}{n!^2\left(\frac{1}{3}\right)_n}$$

$3^{6n}A(n) = 1, 60, 20475, 9373650, 4881796920, \ldots$

- A sequence is P-**recursive** / holonomic if it satisfies a linear recurrence with polynomial coefficients.

EG The **Apéry numbers** $A(n)$ satisfy $A(0) = 1$, $A(1) = 5$ and

$$(n+1)^3 A(n+1) = (2n+1)(17n^2 + 17n + 5)A(n) - n^3 A(n-1).$$

$\zeta(3)$ is irrational!

OPEN Criterion/algorithm for classifying integrality of P-recursive sequences?

- For integral afficionados: (Beukers, '79)

$$(-1)^n \int_0^1 \int_0^1 \int_0^1 \frac{x^n(1-x)^n y^n (1-y)^n z^n (1-z)^n}{(1-(1-xy)z)^{n+1}} \mathrm{d}x\mathrm{d}y\mathrm{d}z = A(n)\zeta(3) - 6B(n)$$

Application: Integrality of P-recursive sequences

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OPEN Criterion/algorithm for classifying integrality of P-recursive sequences?

CONJ Christol '90: Every P-recursive integer sequence of at most exponential growth is the diagonal of a rational function.

EG S 2014: The Apéry numbers are the diagonal of $\dfrac{1}{(1-x-y)(1-z-w)-xyzw}$.

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Applications: asymptotics, congruences, geometry, . . .

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S 2014

The Apéry numbers are the diagonal of $\dfrac{1}{(1-x-y)(1-z-w)-xyzw}$.

- Well-developed theory of multivariate **asymptotics** — e.g., Pemantle–Wilson
- OGFs of such diagonals are algebraic modulo p^r. — Furstenberg, Deligne '67, '84

  Automatically leads to **congruences** such as

$$A(n) \equiv \begin{cases} 1 \pmod{8}, & \text{if } n \text{ even}, \\ 5 \pmod{8}, & \text{if } n \text{ odd}. \end{cases}$$

Chowla–Cowles–Cowles '80
Rowland–Yassawi '13
Rowland–Zeilberger '14

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- Univariate generating function:

$$\sum_{n\geqslant 0} A(n) t^n = \frac{17-t-z}{4\sqrt{2}(1+t+z)^{3/2}} \, {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} \middle| -\frac{1024t}{(1-t+z)^4} \right), \quad z = \sqrt{1-34t+t^2}.$$

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EG constant term — $A(n) = \mathrm{ct}\,[L^n]$ with $L = \dfrac{(1+y)(1+z)(1+x+z)(1+x+z+yz)}{xyz}$

- $F_A(t) = \sum_{n\geqslant 0} A(n)t^n = \mathrm{ct}\left[\dfrac{1}{1-tL}\right]$ is a **period function**.

  The DE satisfied by $F_A(t)$ is the **Picard–Fuchs DE** for the family $V_t : 1-tL=0$.

  Generically, V_t is birationally equivalent to a **K3 surface** with Picard number 19. (Beukers–Peters '84)

Advertisement: Apéry numbers are remarkable

THM Beukers '87

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n\geqslant 0} A(n) \underbrace{\left(\frac{\eta^{12}(\tau)\eta^{12}(6\tau)}{\eta^{12}(2\tau)\eta^{12}(3\tau)}\right)^n}_{\text{modular function}}$$

$1+5q+13q^2+23q^3+O(q^4)$ $\qquad$ $q-12q^2+66q^3+O(q^4)$ $\qquad$ $q=e^{2\pi i\tau}$

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$$A(n) \equiv A(n_0)A(n_1)\cdots A(n_r) \pmod p$$

n_i are the p-adic digits of n

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THM Ahlgren–Ono '00

$$A\left(\frac{p-1}{2}\right) \equiv c(p) \pmod{p^2}$$

$$f(\tau) = \sum_{n\geqslant 1} c(n)q^n = \eta(2\tau)^4\eta(4\tau)^4 \in S_4(\Gamma_0(8))$$





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THM Zagier '16

$$A\left(-\tfrac{1}{2}\right) = \frac{16}{\pi^2} L(f,2)$$

- These extend to **all known** sporadic (Apéry-like) numbers!!!??

! = proven
? = partially known






Lucas congruences: $A(n) \equiv A(n_0) A(n_1) \cdots A(n_r) \pmod{p}$

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THM Malik–S '16: All of the $6+6+3$ known sporadic sequences satisfy Lucas congruences modulo every prime. (Proof long and technical for 2 sequences)

An application of constant term representations

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Suppose the origin is the only interior integral point of the Newton polytope of $P \in \mathbb{Z}[\boldsymbol{x}^{\pm 1}]$.

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THM Gorodetsky '21: Each sporadic sequence, except possibly (η), can be expressed as $\operatorname{ct}[P(\boldsymbol{x})^n]$ so that the result of Samol–van Straten applies.

EG Gorodetsky '21: (η): $\dfrac{(zx+xy-yz-x-1)(xy+yz-zx-y-1)(yz+zx-xy-z-1)}{xyz}$

$(1,0,0)$, $(1,1,0)$ and their permutations are interior points.

Q Algorithmic tools to find (useful) constant term expressions?

Once found, algorithmically provable using creative telescoping.

An application of constant term representations

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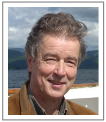

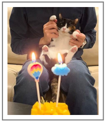

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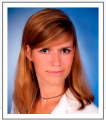

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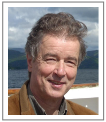

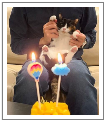

- Beukers–Tsai–Ye (2025) prove Lucas congruences using modular forms.
- Less generically, the sporadic sequences satisfy "super" extensions:

THM S '24: The known sporadic sequences satisfy the **Gessel–Lucas congruences**

$$A(pn+k) \equiv A(k)A(n) + pnA'(k)A(n) \pmod{p^2}.$$

A question of Zagier

- $c(n)$ is a **constant term** if $c(n) = \operatorname{ct}[P^n(\boldsymbol{x})Q(\boldsymbol{x})]$ for Laurent polynomials $P, Q \in \mathbb{Q}[\boldsymbol{x}^{\pm 1}]$ in $\boldsymbol{x} = (x_1, \ldots, x_d)$.

Rowland–Zeilberger '14

EG $Q = 1$

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \operatorname{ct}\left[\left(\frac{(x+y)(z+1)(x+y+z)(y+x+1)}{xyz}\right)^n\right]$$

EG **Catalan**

$$\frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1} = \operatorname{ct}\left[\left(\frac{(x+1)^2}{x}\right)^n (1-x)\right]$$

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Q Zagier '16

Which integer sequences are constant terms?

And in which case can we choose $Q = 1$?

A question of Zagier

- $c(n)$ is a **constant term** if $c(n) = \operatorname{ct}[P^n(\boldsymbol{x})Q(\boldsymbol{x})]$ Rowland–Zeilberger '14
  for Laurent polynomials $P, Q \in \mathbb{Q}[\boldsymbol{x}^{\pm 1}]$ in $\boldsymbol{x} = (x_1, \ldots, x_d)$.

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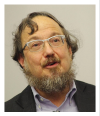

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$$\frac{Q(\boldsymbol{x})}{1 - t x_1 \cdots x_d P(\boldsymbol{x})}$$

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Which diagonals are constant terms?

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- We will answer this in the case of a single variable. (C-finite sequences!)
- For instance: Are Fibonacci numbers constant terms? $\dfrac{x}{1-x-x^2}$

C-finite sequences that are constant terms

- C-finite sequences:

$$\underset{\text{(finite support)}}{A_0(n)} + \sum_{j=1}^{d} \sum_{r=0}^{m_j-1} c_{j,r} n^r \lambda_j^n \qquad \text{(characteristic roots } \lambda_j\text{)}$$

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  And so are sequences of finite support ($\lambda = 0$).

EG $\lambda = 2$

- $2^n = \operatorname{ct}[(x+2)^n] = \operatorname{ct}[2^n]$
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EG Fibonacci numbers are not (sums of) constant terms.

EG $2^n + 1$ is not a constant term but is a sum of two.

Example: Fibonacci numbers

- Our key ingredient to answer these questions are **congruences**:

LEM Bostan, S, Yurkevich '23

If $A(n)$ is a constant term then, for all large enough primes p,

$$A(p) \equiv \underset{\in \mathbb{Q}}{\text{const}} \pmod{p}.$$

proof

$$A(p) = \operatorname{ct}[P(\boldsymbol{x})^p Q(\boldsymbol{x})]$$

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The Fibonacci numbers are $F(n) = \dfrac{\varphi_+^n - \varphi_-^n}{\sqrt{5}}$ with $\varphi_\pm = \dfrac{1 \pm \sqrt{5}}{2}$.

It follows that

$$F(p) \equiv \begin{cases} 1, & \text{if } p \equiv 1, 4 \bmod 5, \\ -1, & \text{if } p \equiv 2, 3 \bmod 5, \end{cases} \pmod{p}.$$

Hence, the Fibonacci numbers cannot be constant terms.

Hypergeometric sequences

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EG open!: Is the following hypergeometric sequence a diagonal?

$$A(n) = \frac{\left(\frac{1}{9}\right)_n \left(\frac{4}{9}\right)_n \left(\frac{5}{9}\right)_n}{n!^2 \left(\frac{1}{3}\right)_n}$$

$3^{6n} A(n) = 1, 60, 20475, 9373650, 4881796920, \ldots$

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LEM Bostan, S, Yurkevich '23 — This hypergeometric sequence is not a constant term (or a linear combination of constant terms).

Proof idea: $A(p)$ takes different values modulo p depending on whether $p \equiv \pm 1 \pmod 9$.

Constant terms are special

- For hypergeometric sequences: (or C-finite or P-recursive)

$$\underset{\text{(or linear combinations)}}{\{\text{constant terms}\}} \subsetneq \{\text{diagonals}\} \subseteq \{P\text{-recursive, globally bounded seq's}\}$$

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- The following is an indication that constant terms are special among diagonals and often have significant additional arithmetic properties.

LEM Bostan, S, Yurkevich '23

Let $A_m(n) = \dfrac{\left(\frac{1}{m}\right)_n \left(1-\frac{1}{m}\right)_n}{n!^2}$ where $m \geqslant 2$ is an integer.

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1. $A_m(n)$ is a diagonal for all $m \geqslant 2$.
2. $A_m(n)$ is a constant term if and only if $m \in \{2, 3, 4, 6\}$.

- The cases $m \in \{2, 3, 4, 6\}$ correspond to the hypergeometric functions underlying Ramanujan's theory of elliptic functions.
  ($m = 2$: classical case; $m = 3, 4, 6$: alternative bases)

Conclusions & Outlook

- Constant terms are an arithmetically interesting subset of diagonals.
- We have classified them in the case of a single variable. Natural classes of sequences to consider next:
  - Hypergeometric sequences
  - Algebraic sequences (diagonals in two variables)
  - Algebraic hypergeometric series
  - Integral factorial ratios (Bober, 2007; via Beukers–Heckman)

EG

Is $A(n) = \frac{(8n)!n!}{(4n)!(3n)!(2n)!} = \binom{8n}{4n}\binom{4n}{n}\binom{2n}{n}^{-1}$ a constant term?

$$1, 140, 60060, 29745716, 15628090140, \ldots = \operatorname{ct}\left[\left(\frac{(1+x)^8}{(1-x)^2x^3}\right)^n\right]$$

This is algebraic (and therefore a diagonal) and hypergeometric.

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This is algebraic (and therefore a diagonal) and hypergeometric.

- How to find representations as (nice) constant terms or diagonals?
  Once found, such representations can be proved using **creative telescoping**.
- How unique are the Laurent polynomials in a constant term?
  Connections to cluster algebras, mutations of Laurent polynomials, ...

THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`

A. Bostan, A. Straub, S. Yurkevich
On the representability of sequences as constant terms
Journal of Number Theory, Vol. 253, 2023, p. 235–256

A. Straub
Gessel–Lucas congruences for sporadic sequences
Monatshefte für Mathematik, Vol. 203, 2024, p. 883-898