

Interpolated sequences and critical L -values of modular forms

Southern Regional Number Theory Conference:
Modular Curves, Modular Forms, and Hypergeometric Functions
LSU

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$$A(n) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 \quad f(\tau) = \eta(2\tau)^4 \eta(4\tau)^4 = \sum_{n \geq 1} \alpha_n q^n$$

1, 5, 73, 1445, 33001, 819005, 21460825, ...

$$A\left(\frac{p-1}{2}\right) \equiv \alpha_p \pmod{p^2}$$

$$A\left(-\frac{1}{2}\right) = \frac{16}{\pi^2} L(f, 2)$$



Joint work with:

Robert Osburn
(University College Dublin)



Assorted background

The Riemann zeta function

- The **Riemann zeta function** is the analytic continuation of

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

- Its zeros and values are fundamental, yet mysterious to this day.

CONJ
RH

If $\zeta(s) = 0$ then $s \in \{-2, -4, \dots\}$ or $\operatorname{Re}(s) = \frac{1}{2}$.

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THM
Euler
1734

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \dots, \quad \zeta(2n) = \frac{(-1)^{n+1} (2\pi)^{2n} B_{2n}}{2(2n)!}$$

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$\pi, \zeta(3), \zeta(5), \dots$ are algebraically independent over \mathbb{Q} .

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THM
Apéry '78

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- Open: $\zeta(5)$ is irrational
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- Open: $\zeta(3)/\pi^3$ is irrational
- Open: Catalan's constant $G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$ is irrational

Values of modular forms and functions at CM points

THM
Chowla–
Selberg
1967

$$\prod_{j=1}^h a_j^{-6} |\eta(\tau_j)|^{24} = \frac{1}{(2\pi d)^{6h}} \left[\prod_{k=1}^d \Gamma\left(\frac{k}{d}\right)^{\left(\frac{-d}{k}\right)} \right]^{3w}$$

where the product is over reduced binary quadratic forms $[a_j, b_j, c_j]$ of discriminant $-d < 0$.

$$\tau_j = \frac{-b_j + \sqrt{-d}}{2a_j}$$

throughout, $-d$ is a discriminant; w is number of roots of unity in $\mathbb{Q}(\sqrt{-d})$, which is $w = 2$ except for $d = 3, 4$

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EG

$\mathbb{Q}(\sqrt{-15})$ has discriminant $-d = -15$ and class number $h = 2$.

$$\begin{aligned} Q_1 &= [1, 1, 4] & Q_2 &= [2, 1, 2] \\ \tau_1 &= -\frac{1}{2} + \frac{1}{2}\sqrt{-15}, & \tau_2 &= \frac{1}{2}\tau_1 \end{aligned}$$

$$\begin{aligned} \frac{1}{\sqrt{2}} |\eta(\tau_1)\eta(\tau_2)|^2 &= \frac{1}{30\pi} \left(\frac{\Gamma(\frac{1}{15})\Gamma(\frac{2}{15})\Gamma(\frac{4}{15})\Gamma(\frac{8}{15})}{\Gamma(\frac{7}{15})\Gamma(\frac{11}{15})\Gamma(\frac{13}{15})\Gamma(\frac{14}{15})} \right)^{1/2} \\ &= \frac{1}{120\pi^3} \Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right) \end{aligned}$$

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LEM If $\sigma_1, \sigma_2 \in \mathcal{H} \cap \mathbb{Q}(\sqrt{-d})$, then $\frac{\eta(\sigma_1)}{\eta(\sigma_2)}$ is algebraic.

proof

- $\sigma_2 = M \cdot \sigma_1$ and $\sigma_1 = N \cdot \sigma_1$ for some $M, N \in \text{GL}_2(\mathbb{Z})$. [$M \neq \text{id}$]
- $f(\tau) = \frac{\eta(\tau)}{\eta(M \cdot \tau)}$ and $f(N \cdot \tau)$ are modular functions.

□

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 - There is an algebraic relation $\Phi(f(\tau), f(N \cdot \tau)) = 0$.
 - Then: $\Phi(f(\sigma_1), f(\sigma_1)) = \Phi\left(\frac{\eta(\sigma_1)}{\eta(\sigma_2)}, \frac{\eta(\sigma_1)}{\eta(\sigma_2)}\right) = 0$ □

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THM

For each $\mathbb{Q}(\sqrt{-d})$, let $\omega_d = \frac{1}{\pi^{h/2}} \left[\prod_{k=1}^d \Gamma\left(\frac{k}{d}\right)^{\binom{-d}{k}} \right]^{w/4}$.

For any weight k modular form $f(\tau)$ and any $\sigma \in \mathcal{H} \cap \mathbb{Q}(\sqrt{-d})$, we have $f(\sigma) \in \omega_d^k \bar{\mathbb{Q}}$.

[assuming the functions have algebraic Fourier coefficients]

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EG

$$\eta(i) = \frac{1}{2\pi^{3/4}} \Gamma\left(\frac{1}{4}\right)$$

$$\theta_3(i) = \frac{1}{\sqrt{2}\pi^{3/4}} \Gamma\left(\frac{1}{4}\right)$$

$$\theta_3(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2/2} = \frac{\eta(\tau)^5}{\eta(\tau/2)^2 \eta(2\tau)^2}$$

$$\theta_3(1 + i\sqrt{2})^4 = \frac{\Gamma^2\left(\frac{1}{8}\right) \Gamma^2\left(\frac{3}{8}\right)}{8\sqrt{2}\pi^3}$$

$$\theta_3\left(-\frac{1-i\sqrt{3}}{2}\right)^4 = \frac{(3 - i\sqrt{3}) \Gamma^6\left(\frac{1}{3}\right)}{2^{11/3} \pi^4}.$$

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Question:

CAS implementation?

(We have efficient symbolic-numerical algorithms for values of specific modular functions like j and λ .)



Apéry-like sequences

- The **Apéry numbers**

1, 5, 73, 1445, ...

$$A(n) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 A(n+1) = (2n+1)(17n^2 + 17n + 5)A(n) - n^3 A(n-1).$$

Apéry numbers and the irrationality of $\zeta(3)$

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THM Apéry '78 $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.

proof The same recurrence is satisfied by the “near”-integers

$$B(n) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^n \frac{1}{j^3} + \sum_{m=1}^k \frac{(-1)^{m-1}}{2m^3 \binom{n}{m} \binom{n+m}{m}} \right).$$

Then, $\frac{B(n)}{A(n)} \rightarrow \zeta(3)$. But too fast for $\zeta(3)$ to be rational. \square

Goal: a recurrence for $\sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 =: \sum_{k=0}^n A(n, k)$

Let S_n be such that $S_n f(n, k) = f(n+1, k)$.



Marko Petkovsek, Herbert S. Wilf and Doron Zeilberger

A = B

A. K. Peters, Ltd., 1st edition, 1996

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- Suppose we have $P(n, S_n) \in \mathbb{Q}[n, S_n]$ and $R(n, k) \in \mathbb{Q}(n, k)$ so that

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- Then: $P(n, S_n) \sum_{k \in \mathbb{Z}} A(n, k) = 0$

EG

$$P(n, S_n) = (n+2)^3 S_n^2 - (2n+3)(17n^2 + 51n + 39)S_n + (n+1)^3$$
$$R(n, k) = \frac{4k^4(2n+3)(4n^2 - 2k^2 + 12n + 3k + 8)}{(n-k+1)^2(n-k+2)^2}$$

Automatically obtained using Koutschan's excellent **HolonomicFunctions** package for Mathematica.



Marko Petkovsek, Herbert S. Wilf and Doron Zeilberger

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Zagier's search and Apéry-like numbers

- Recurrence for Apéry numbers is the case $(a, b, c) = (17, 5, 1)$ of

$$(n + 1)^3 u_{n+1} = (2n + 1)(an^2 + an + b)u_n - cn^3 u_{n-1}.$$

Q
Beukers,
Zagier

Are there other tuples (a, b, c) for which the solution defined by $u_{-1} = 0, u_0 = 1$ is integral?

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Are there other tuples (a, b, c) for which the solution defined by $u_{-1} = 0, u_0 = 1$ is integral?

- Essentially, only 14 tuples (a, b, c) found. (Almkvist–Zudilin)
 - 4 hypergeometric and 4 Legendrian solutions (with generating functions

$${}_3F_2 \left(\begin{matrix} \frac{1}{2}, \alpha, 1-\alpha \\ 1, 1 \end{matrix} \middle| 4C_\alpha z \right), \quad \frac{1}{1-C_\alpha z} {}_2F_1 \left(\begin{matrix} \alpha, 1-\alpha \\ 1 \end{matrix} \middle| \frac{-C_\alpha z}{1-C_\alpha z} \right)^2,$$

with $\alpha = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$ and $C_\alpha = 2^4, 3^3, 2^6, 2^4 \cdot 3^3$

- 6 sporadic solutions
- Similar (and intertwined) story for:
 - $(n+1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2 u_{n-1}$ (Beukers, Zagier)
 - $(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - n(cn^2 + d)u_{n-1}$ (Cooper)

The six sporadic Apéry-like numbers

(a, b, c)	$A(n)$	
$(17, 5, 1)$	$\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$	Apéry numbers
$(12, 4, 16)$	$\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$	
$(10, 4, 64)$	$\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$	Domb numbers
$(7, 3, 81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$	Almkvist-Zudilin numbers
$(11, 5, 125)$	$\sum_k (-1)^k \binom{n}{k}^3 \binom{4n-5k}{3n}$	
$(9, 3, -27)$	$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$	

Modularity of Apéry-like numbers

- The Apéry numbers

1, 5, 73, 1145, ...

satisfy

$$A(n) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2$$

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n \geq 0} A(n) \underbrace{\left(\frac{\eta^{12}(\tau)\eta^{12}(6\tau)}{\eta^{12}(2\tau)\eta^{12}(3\tau)} \right)^n}_{\text{modular function}} .$$

$$1 + 5q + 13q^2 + 23q^3 + O(q^4)$$

$$q - 12q^2 + 66q^3 + O(q^4)$$

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$$q - 12q^2 + 66q^3 + O(q^4)$$

FACT Not at all evidently, such a **modular parametrization** exists for all known Apéry-like numbers!

- Context:
 - $f(\tau)$ modular form of weight k
 - $x(\tau)$ modular function
 - $y(x)$ such that $y(x(\tau)) = f(\tau)$

Then $y(x)$ satisfies a linear differential equation of order $k + 1$.

Supercongruences for Apéry numbers

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THM

Beukers,
Coster
'85, '88

The Apéry numbers satisfy the **supercongruence** $(p \geq 5)$

$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}.$$

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EG

For primes p , simple combinatorics proves the congruence

$$\binom{2p}{p} = \sum_k \binom{p}{k} \binom{p}{p-k} \equiv 1 + 1 \pmod{p^2}.$$

For $p \geq 5$, Wolstenholme's congruence shows that, in fact,

$$\binom{2p}{p} \equiv 2 \pmod{p^3}.$$

Supercongruences for Apéry-like numbers



Robert Osburn
(University of Dublin)



Brundaban Sahu
(NISER, India)

- Conjecturally, supercongruences like

$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}$$

hold for all Apéry-like numbers.

Osburn–Sahu '09

- Current state of affairs for the six sporadic sequences from earlier:

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$(7, 3, 81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$	open modulo p^3 Amdeberhan–Tauraso '16
$(11, 5, 125)$	$\sum_k (-1)^k \binom{n}{k}^3 \binom{4n-5k}{3n}$	Osburn–Sahu–S '16
$(9, 3, -27)$	$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$	Gorodetsky '18



Interpolations and critical *L*-values

The Golyshev–Zagier observation

THM
Ahlgren–
Ono
'00

For any odd prime p , the Apéry numbers for $\zeta(3)$ satisfy

$$A\left(\frac{p-1}{2}\right) \equiv a_f(p) \pmod{p^2},$$

with $f(\tau) = \eta(2\tau)^4 \eta(4\tau)^4 = \sum_{n \geq 1} a_f(n) q^n \in S_4(\Gamma_0(8))$.

conjectured (and proved modulo p) by Beukers '87

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THM
Zagier
'16

$$A\left(-\frac{1}{2}\right) = \frac{16}{\pi^2} L(f, 2)$$

- Here, $A(x) = \sum_{k=0}^{\infty} \binom{x}{k}^2 \binom{x+k}{k}^2$ is absolutely convergent for $x \in \mathbb{C}$.
- Predicted by Golyshev based on motivic considerations, the connection of the Apéry numbers with the double covering of a family of K3 surfaces, and the Tate conjecture.

Zagier's six sporadic sequences

$$(n+1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2 u_{n-1}$$

*	(a, b, c)	$C_*(n)$	
A	$(7, 2, -8)$	$\sum_{k=0}^n \binom{n}{k}^3$	Franl numbers
B	$(9, 3, 27)$	$\sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k 3^{n-3k} \binom{n}{3k} \frac{(3k)!}{k!^3}$	
C	$(10, 3, 9)$	$\sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k}$	
D	$(11, 3, -1)$	$\sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{n}$	Apéry numbers
E	$(12, 4, 32)$	$\sum_{k=0}^n \binom{n}{k} \binom{2k}{k} \binom{2(n-k)}{n-k}$	
F	$(17, 6, 72)$	$\sum_{k=0}^n (-1)^k 8^{n-k} \binom{n}{k} C_A(k)$	

Congruences and L -valued interpolations

- For $*$ one of A - F , let $C_*(n)$ be Zagier's sporadic sequence.

THM

Beukers,
Stienstra
'85; OS
'18;
Kazalicki
'18

For $*$ one of A - F , there exists a newform $f_*(\tau) = \sum_{n \geq 1} \gamma_{n,*} q^n$ of weight 3, such that, for all primes $p > 2$,

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For $*$ one of A - F , except E , there is $\alpha_* \in \mathbb{Z}$ such that

$$C_*\left(-\frac{1}{2}\right) = \frac{\alpha_*}{\pi^2} L(f_*, 2).$$

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For sequence E ,

$$\operatorname{res}_{x=-1/2} C_E(x) = \frac{6}{\pi^2} L(f_E, 1).$$

Interpolations of Zagier's six sporadic sequences

*	$C_*(n)$	$C_*(x)$
A	$\sum_{k=0}^n \binom{n}{k}^3$	$\sum_{k \geq 0} \binom{x}{k}^3$
B	$\sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k 3^{n-3k} \binom{n}{3k} \frac{(3k)!}{k!^3}$	$\sum_{k \geq 0} (-1)^k 3^{x-3k} \binom{x}{3k} \frac{(3k)!}{k!^3}$
C	$\sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k}$	$\operatorname{Re} {}_3F_2 \left(\begin{matrix} -x, -x, 1/2 \\ 1, 1 \end{matrix} \middle 4 \right)$
D	$\sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{n}$	$\sum_{k \geq 0} \binom{x}{k}^2 \binom{x+k}{x}$
E	$\sum_{k=0}^n \binom{n}{k} \binom{2k}{k} \binom{2(n-k)}{n-k}$	$\sum_{k \geq 0} \binom{x}{k} \binom{2k}{k} \binom{2(x-k)}{x-k}$
F	$\sum_{k=0}^n (-1)^k 8^{n-k} \binom{n}{k} C_A(k)$	$\sum_{k \geq 0} (-1)^k 8^{x-k} \binom{x}{k} C_A(k)$

The weight 3 newforms of level N_* and their L -values

$$C_*(-\frac{1}{2}) = \frac{\alpha_*}{\pi^2} L(f_*, 2)$$

$*$	$f_*(\tau)$	N_*	CM	$L(f_*, 2)$	α_*
A	$\frac{\eta(4\tau)^5 \eta(8\tau)^5}{\eta(2\tau)^2 \eta(16\tau)^2}$	32	$\mathbb{Q}(\sqrt{-2})$	$\frac{\Gamma^2(\frac{1}{8}) \Gamma^2(\frac{3}{8})}{64\sqrt{2}\pi}$	8
B	$\eta(4\tau)^6$	16	$\mathbb{Q}(\sqrt{-1})$	$\frac{\Gamma^4(\frac{1}{4})}{64\pi}$	8
C	$\eta(2\tau)^3 \eta(6\tau)^3$	12	$\mathbb{Q}(\sqrt{-3})$	$\frac{\Gamma^6(\frac{1}{3})}{2^{17/3}\pi^2}$	12
D	$\eta(4\tau)^6$	16	$\mathbb{Q}(\sqrt{-1})$	$\frac{\Gamma^4(\frac{1}{4})}{64\pi}$	16
E	$\eta(\tau)^2 \eta(2\tau) \eta(4\tau) \eta(8\tau)^2$	8	$\mathbb{Q}(\sqrt{-2})$	$\frac{\Gamma^2(\frac{1}{8}) \Gamma^2(\frac{3}{8})}{192\pi}$	6
F	$q - 2q^2 + 3q^3 + \dots$	24	$\mathbb{Q}(\sqrt{-6})$	$\frac{\Gamma(\frac{1}{24}) \Gamma(\frac{5}{24}) \Gamma(\frac{7}{24}) \Gamma(\frac{11}{24})}{96\sqrt{6}\pi}$	6

THM
Ribet
'76

A newform has CM by a quadratic field K (necessarily imaginary and unique) if and only if it comes from a Hecke character of K .

- L -values can then be approached using the work of Damerell ('70)
- L -values for A - E computed by Rogers, Wan and Zucker ('15) using binary theta series
- More recent work on explicitly evaluating L -values of CM modular forms by Li, Long, Tu ('18)



W.-C. W. Li, L. Long and F.-T. Tu

Computing special L -values of certain modular forms with complex multiplication
SIGMA, 14(090), 2018, p. 1–32



M. Rogers, J. G. Wan and I. J. Zucker

Moments of elliptic integrals and critical L -values
Ramanujan Journal, 37, 2015, p. 113–130

Congruences and interpolations for cellular integrals

- For an explicit family σ_N of convergent configurations,
 $A_{\sigma_N}(n) = C_D(n)^{(N-3)/2}$.

More shortly!

Congruences and interpolations for cellular integrals

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More shortly!

- For odd $k \geq 3$, consider the weight k binary theta series

$$f_k(\tau) = \frac{1}{4} \sum_{(n,m) \in \mathbb{Z}^2} (-1)^{m(k-1)/2} (n - im)^{k-1} q^{n^2+m^2} =: \sum_{n \geq 1} \gamma_k(n) q^n.$$

Congruences and interpolations for cellular integrals

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THM

McCarthy,
OS '18

Let $N \geq 5$ be odd and $k = N - 2$. Then, for all primes $p \geq 5$,

$$A_{\sigma_N}\left(\frac{p-1}{2}\right) \equiv \gamma_k(p) \pmod{p^2}.$$

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THM

OS '18

Let $N \geq 5$ be odd and $k = N - 2$. Then,

$$A_{\sigma_N}\left(-\frac{1}{2}\right) = \frac{\alpha_k}{\pi^{k-1}} L(f_k, k-1),$$

where α_k are explicit rational numbers, defined recursively.

IV

Selected details and proof sketches

Proof of an interpolation: a simple case

LEM

$$C_D(-\frac{1}{2}) = \frac{16}{\pi^2} L(f_D, 2)$$

proof

Here,
$$C_D(x) = \sum_{k \geq 0} \binom{x}{k}^2 \binom{x+k}{x}$$



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Here, $C_D(x) = \sum_{k \geq 0} \binom{x}{k}^2 \binom{x+k}{x} = {}_3F_2 \left(\begin{matrix} -x, -x, x+1 \\ 1, 1 \end{matrix} \middle| 1 \right)$.



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Using hypergeometric identities,

$$C_D(-\frac{1}{2}) = {}_3F_2 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} \middle| 1 \right) = \frac{\Gamma^4(\frac{1}{4})}{4\pi^3}.$$

It remains to show $L(f_D, 2) = \frac{\Gamma^4(\frac{1}{4})}{64\pi}$.

[Rogers, Wan and Zucker ('15)] □

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□

- For more challenging cases, a modular parametrization is crucial:

$${}_3F_2 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} \middle| 4\lambda(\tau)(1 - \lambda(\tau)) \right) = {}_2F_1 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \middle| \lambda(\tau) \right)^2 = \theta_3(\tau)^4$$

- For $\tau = i$, we get $\lambda(i) = \frac{1}{2}$ and $\theta_3(i)^2 = \frac{\Gamma^2(1/4)}{2\pi^{3/2}}$.

Proof another interpolation: tricky case

LEM
Zudilin
'18

$$C_{\mathbf{F}}(-\frac{1}{2}) = \frac{6}{\pi^2} L(f_{\mathbf{F}}, 2)$$

- Here, $C_{\mathbf{F}}(-\frac{1}{2}) = \frac{1}{\sqrt{8}} \sum_{k=0}^{\infty} 2^{-5k} \binom{2k}{k} C_{\mathbf{A}}(k)$

Proof another interpolation: tricky case

LEM
Zudilin
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$$C_{\mathbf{F}}(-\tfrac{1}{2}) = \frac{6}{\pi^2} L(f_{\mathbf{F}}, 2) = \frac{\Gamma(\frac{1}{24})\Gamma(\frac{5}{24})\Gamma(\frac{7}{24})\Gamma(\frac{11}{24})}{16\sqrt{6}\pi^3}$$

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$$g(z) = \sum_{k=0}^{\infty} z^k \binom{2k}{k} \sum_{j=0}^k \binom{k}{j}^3$$

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has the modular parametrization [Chan, Tanigawa, Yang, Zudilin ('11)]

$$g\left(\frac{x(\tau)}{(1-x(\tau))^2}\right) = \frac{1}{6}(6E_2(6\tau) + 3E_2(3\tau) - 2E_2(2\tau) - E_2(\tau)),$$

with

$$x(\tau) = \left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)}\right)^{12}, \quad E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}.$$

Proof another interpolation: tricky case

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Zudilin
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$$C_{\mathbf{F}}(-\tfrac{1}{2}) = \frac{6}{\pi^2} L(f_{\mathbf{F}}, 2) = \frac{\Gamma(\frac{1}{24})\Gamma(\frac{5}{24})\Gamma(\frac{7}{24})\Gamma(\frac{11}{24})}{16\sqrt{6}\pi^3}$$

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- Setting $\tau = i/\sqrt{6}$ then leads to the result.

(On the other hand, $L(f_{\mathbf{F}}, s)$ is a Hecke L -series on the field $\mathbb{Q}(\sqrt{-6})$.)

V

Brown's cellular integrals

$$I_n = (-1)^n \int_0^1 \int_0^1 \frac{x^n(1-x)^n y^n(1-y)^n}{(1-xy)^{n+1}} dx dy$$

$$J_n = \frac{1}{2} \int_0^1 \int_0^1 \int_0^1 \frac{x^n(1-x)^n y^n(1-y)^n w^n(1-w)^n}{(1-(1-xy)w)^{n+1}} dx dy dw$$

- Beukers showed that

$$I_n = a(n)\zeta(2) + \tilde{a}(n), \quad J_n = b(n)\zeta(3) + \tilde{b}(n)$$

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where $\tilde{a}(n), \tilde{b}(n) \in \mathbb{Q}$ and

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- Brown realizes these as period integrals, for $N = 5, 6$, on the moduli space $\mathcal{M}_{0,N}$ of curves of genus 0 with N marked points.

THM
Brown
2009

Period integrals on $\mathcal{M}_{0,N}$ are \mathbb{Q} -linear combinations of multiple zeta values (MZVs).
(conjectured by Goncharov–Manin, 2004)

- Examples of such integrals can be written as: $(a_i, b_j, c_{ij} \in \mathbb{Z})$

$$\int_{0 < t_1 < \dots < t_{N-3} < 1} \prod t_i^{a_i} (1 - t_j)^{b_j} (t_i - t_j)^{c_{ij}} dt_1 \dots dt_{N-3}$$

- Typically involve MZVs of all weights $\leq N - 3$.

Brown's cellular integrals

THM
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- Typically involve MZVs of all weights $\leq N - 3$.
- Brown constructs families of integrals $I_\sigma(n)$, for which MZVs of submaximal weight vanish.

Here, σ are certain (“convergent”) permutations in S_N .

N	5	6	7	8	9	10	11
# of σ	1	1	5	17	105	771	7028

One of Brown's cellular integrals

- One of the 17 permutations for $N = 8$ is $\sigma = (8, 3, 6, 1, 4, 7, 2, 5)$.
- Cellular integral $I_\sigma(n) = \int_\Delta f_\sigma^n \omega_\sigma$ where $\Delta : 0 < t_2 < \dots < t_6 < 1$

$$f_\sigma = \frac{(-t_2)(t_2 - t_3)(t_3 - t_4)(t_4 - t_5)(t_5 - t_6)(t_6 - 1)}{(t_3 - t_6)(t_6)(-t_4)(t_4 - 1)(1 - t_2)(t_2 - t_5)}, \quad \omega_\sigma = \frac{dt_2 dt_3 dt_4 dt_5 dt_6}{(t_3 - t_6)(t_6)(-t_4)(t_4 - 1)(1 - t_2)(t_2 - t_5)}.$$

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$$f_\sigma = \frac{(-t_2)(t_2 - t_3)(t_3 - t_4)(t_4 - t_5)(t_5 - t_6)(t_6 - 1)}{(t_3 - t_6)(t_6)(-t_4)(t_4 - 1)(1 - t_2)(t_2 - t_5)}, \quad \omega_\sigma = \frac{dt_2 dt_3 dt_4 dt_5 dt_6}{(t_3 - t_6)(t_6)(-t_4)(t_4 - 1)(1 - t_2)(t_2 - t_5)}.$$

EG
Panzer:
HyperInt

$$I_\sigma(0) = 16\zeta(5) - 8\zeta(3)\zeta(2)$$

$$I_\sigma(1) = 33I_\sigma(0) - 432\zeta(3) + 316\zeta(2) - 26$$

$$I_\sigma(2) = 8929I_\sigma(0) - 117500\zeta(3) + \frac{515189}{6}\zeta(2) - \frac{331063}{48}$$

One of Brown's cellular integrals

- One of the 17 permutations for $N = 8$ is $\sigma = (8, 3, 6, 1, 4, 7, 2, 5)$.
- Cellular integral $I_\sigma(n) = \int_\Delta f_\sigma^n \omega_\sigma$ where $\Delta : 0 < t_2 < \dots < t_6 < 1$

$$f_\sigma = \frac{(-t_2)(t_2 - t_3)(t_3 - t_4)(t_4 - t_5)(t_5 - t_6)(t_6 - 1)}{(t_3 - t_6)(t_6)(-t_4)(t_4 - 1)(1 - t_2)(t_2 - t_5)}, \quad \omega_\sigma = \frac{dt_2 dt_3 dt_4 dt_5 dt_6}{(t_3 - t_6)(t_6)(-t_4)(t_4 - 1)(1 - t_2)(t_2 - t_5)}.$$

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With 2-dimensional space of analytic solutions at 0.

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With 2-dimensional space of analytic solutions at 0.
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$$1, 33, 8929, 4124193, 2435948001, 1657775448033, \dots$$

One of Brown's cellular integrals, cont'd

- One of the 17 permutations for $N = 8$ is $\sigma = (8, 3, 6, 1, 4, 7, 2, 5)$.
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- The leading coefficients $A_\sigma(n)$ of $I_\sigma(n)$ are:

1, 33, 8929, 4124193, 2435948001, 1657775448033, ...

LEM
McCarthy,
Osburn,
S 2018

$$A_\sigma(n) = \sum_{\substack{k_1, k_2, k_3, k_4=0 \\ k_1+k_2=k_3+k_4}}^n \prod_{i=1}^4 \binom{n}{k_i} \binom{n+k_i}{k_i}$$

One of Brown's cellular integrals, cont'd

- One of the 17 permutations for $N = 8$ is $\sigma = (8, 3, 6, 1, 4, 7, 2, 5)$.
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LEM
McCarthy,
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CONJ
McCarthy,
Osburn,
S 2018

For each $N \geq 5$ and convergent σ_N , the leading coefficients $A_{\sigma_N}(n)$ satisfy ($p \geq 5$)

$$A_{\sigma_N}(mp^r) \equiv A_{\sigma_N}(mp^{r-1}) \pmod{p^{3r}}.$$

For $N = 5, 6$ these are the supercongruences proved by Beukers and Coster.

One of Brown's cellular integrals, cont'd

- One of the 17 permutations for $N = 8$ is $\sigma = (8, 3, 6, 1, 4, 7, 2, 5)$.
- Cellular integral $I_\sigma(n) = \int_\Delta f_\sigma^n \omega_\sigma$ where
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LEM
McCarthy,
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THM
McCarthy,
Osburn,
S 2018

For any odd prime p ,

$$A_\sigma\left(\frac{p-1}{2}\right) \equiv \gamma(p) \pmod{p^2}.$$

where $\eta^{12}(2\tau) = \sum_{n \geq 1} \gamma(n)q^n$ is the unique newform in $S_6(\Gamma_0(4))$.

The Ahlgren–Ono supercongruences

THM
Ahlgren–
Ono
'00

For any odd prime p , the Apéry numbers for $\zeta(3)$ satisfy

$$A\left(\frac{p-1}{2}\right) \equiv \alpha(p) \pmod{p^2},$$

with $\eta(2\tau)^4\eta(4\tau)^4 = \sum_{n \geq 1} \alpha(n)q^n$ the unique newform in $S_4(\Gamma_0(8))$.

THM
Ahlgren
'01

For any prime $p \geq 5$, the Apéry numbers for $\zeta(2)$ satisfy

$$B\left(\frac{p-1}{2}\right) \equiv \beta(p) \pmod{p^2},$$

with $\eta(4\tau)^6 = \sum_{n \geq 1} \beta(n)q^n$ the unique newform in $S_3(\Gamma_0(16), (\frac{-4}{\cdot}))$.

- conjectured (and proved modulo p) by Beukers '87

An infinite family of supercongruences

- $A_{\sigma_N}(n) = B(n)^{(N-3)/2}$ is one of Brown's sequences for a certain σ_N . Here, $B(n)$ are the Apéry numbers for $\zeta(2)$.
- For odd $k \geq 3$, consider the weight k binary theta series

$$f_k(\tau) = \frac{1}{4} \sum_{(n,m) \in \mathbb{Z}^2} (-1)^{m(k-1)/2} (n - im)^{k-1} q^{n^2+m^2} = \sum_{n \geq 1} \gamma_k(n) q^n.$$

THM
McCarthy,
Osburn,
S 2018

Let $N \geq 5$ be odd. For any prime $p \geq 5$,

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THM
McCarthy,
Osburn,
S 2018

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- Q** Supercongruences for all of Brown's sequences?
Maybe arising from L -series attached to Galois representations?

$${}_4F_3 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1 \end{matrix} \middle| 1 \right)_{p-1} \equiv \alpha(p) \pmod{p^3}, \quad (p \geq 3)$$

with $\eta(2\tau)^4 \eta(4\tau)^4 = \sum_{n \geq 1} \alpha(n) q^n$ the unique newform in $S_4(\Gamma_0(8))$.

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- This result proved the first of 14 related supercongruences conjectured by Rodriguez-Villegas (2001) between
 - truncated hypergeometric series ${}_4F_3$ and
 - Fourier coefficients of modular forms of weight 4.
- 11 of these remained open until recently proved by Long, Tu, Yui, Zudilin (2017).

McCarthy (2010), Fuselier–McCarthy (2016) prove one each; McCarthy (2010) proves “half” of each of the 14.

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Q Can the supercongruences for Brown’s sequences be similarly embedded in the hypergeometric setting?

A supercongruence for ${}_6F_5$

THM

Osburn,
S, Zudilin
2018

$${}_6F_5 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1, 1, 1 \end{matrix} \middle| 1 \right)_{p-1} \equiv \lambda(p) \pmod{p^3},$$

for primes $p > 2$. Here, $\lambda(n)$ are the Fourier coefficients of

$$\eta(\tau)^8 \eta(4\tau)^4 + 8\eta(4\tau)^{12} = \sum_{n \geq 1} \lambda(n) q^n \in S_6(\Gamma_0(8)).$$

- Conjectured by Mortenson based on numerical evidence, which further suggests it holds modulo p^5 .

A supercongruence for ${}_6F_5$

THM

Osburn,
S, Zudilin
2018

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- Conjectured by Mortenson based on numerical evidence, which further suggests it holds modulo p^5 .
- A result of Frechette, Ono and Papanikolas expresses the $\lambda(p)$ in terms of Gaussian hypergeometric functions.
- Osburn and Schneider determined the resulting Gaussian hypergeometric functions modulo p^3 in terms of sums involving harmonic sums.

A supercongruence for ${}_6F_5$

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Osburn,
S, Zudilin
2018

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Q Why do these supercongruences hold?

Promising explanation suggested by Roberts, Rodriguez-Villegas (2017) in terms of gaps between Hodge numbers of an associated motive.

VI

L-value interpolations for cellular integrals

Congruences and interpolations for cellular integrals

- For an explicit family σ_N of convergent configurations,
 $A_{\sigma_N}(n) = C_D(n)^{(N-3)/2}$.
- For odd $k \geq 3$, consider the weight k binary theta series

$$f_k(\tau) = \frac{1}{4} \sum_{(n,m) \in \mathbb{Z}^2} (-1)^{m(k-1)/2} (n - im)^{k-1} q^{n^2+m^2} =: \sum_{n \geq 1} \gamma_k(n) q^n.$$

THM
McCarthy,
OS '18

Let $N \geq 5$ be odd and $k = N - 2$. Then, for all primes $p \geq 5$,

$$A_{\sigma_N}\left(\frac{p-1}{2}\right) \equiv \gamma_k(p) \pmod{p^2}.$$

THM
OS '18

Let $N \geq 5$ be odd and $k = N - 2$. Then,

$$A_{\sigma_N}\left(-\frac{1}{2}\right) = \frac{\alpha_k}{\pi^{k-1}} L(f_k, k-1),$$

where α_k are explicit rational numbers, defined recursively.

THM
OS '18

Let $N \geq 5$ be odd and $k = N - 2$. Then,

$$A_{\sigma_N}(-\frac{1}{2}) = \frac{\alpha_k}{\pi^{k-1}} L(f_k, k-1),$$

where α_k are given as follows:

$$\alpha_k = 2^{(k+1)/2}(k-2) \begin{cases} 2/r_{(k-1)/2}, & \text{if } k \equiv 1 \pmod{4}, \\ 1/s_{(k-1)/2}, & \text{if } k \equiv 3 \pmod{4}. \end{cases}$$

Here, r_n is defined by $r_2 = 1/5$, $r_3 = 0$ and

$$(2n+1)(n-3)r_n = 3 \sum_{k=2}^{n-2} r_k r_{n-k}$$

for $n \geq 4$, and s_n is defined by $s_1 = 1/4$, $s_2 = 11/80$, $s_3 = 1/32$ and the same recursion for $n \geq 4$.

Interpolations for cellular integrals

THM
OS '18

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for $n \geq 4$, and s_n is defined by $s_1 = 1/4$, $s_2 = 11/80$, $s_3 = 1/32$ and the same recursion for $n \geq 4$.

- Since $A_{\sigma_N}(-\frac{1}{2}) = \left(\frac{\Gamma^2(\frac{1}{4})}{2\pi^{3/2}}\right)^{N-3}$, it remains to evaluate $L(f_k, k-1)$.

Evaluating $L(f_k, k - 1)$

$$f_k(\tau) = \frac{1}{4} \sum_{(n,m) \in \mathbb{Z}^2} (-1)^{m(k-1)/2} (n - im)^{k-1} q^{n^2+m^2}$$

- The L -value of interest is:

$$L(f_k, k - 1) = \frac{1}{4} \sum_{(n,m) \neq (0,0)} (-1)^{m(k-1)/2} \frac{(n - im)^{k-1}}{(n^2 + m^2)^{k-1}}$$

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$$G_\ell(\tau) = \sum_{(n,m) \neq (0,0)} \frac{1}{(n + m\tau)^\ell}$$

$$= \frac{1}{4} \sum_{(n,m) \neq (0,0)} (-1)^{m(k-1)/2} \frac{1}{(n + im)^{k-1}}$$

$$= \frac{1}{4} \begin{cases} G_{k-1}(i), & \text{if } k \equiv 1 \pmod{4}, \\ 2G_{k-1}(2i) - G_{k-1}(i), & \text{if } k \equiv 3 \pmod{4}. \end{cases}$$

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- For $n \geq 4$,

$$(4n^2 - 1)(n - 3)G_{2n} = 3 \sum_{k=2}^{n-2} (2k - 1)(2n - 2k - 1)G_{2k}G_{2(n-k)}.$$

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$$(4n^2 - 1)(n - 3)G_{2n} = 3 \sum_{k=2}^{n-2} (2k - 1)(2n - 2k - 1)G_{2k}G_{2(n-k)}.$$

- It remains to evaluate $G_\ell(\tau)$ for $\ell \in \{4, 6\}$ and $\tau \in \{i, 2i\}$.

All critical L -values of f_k

EG
Empirically

$$L(f_5, 4) = \frac{2\pi}{5} L(f_5, 3) = \frac{\pi^2}{5} L(f_5, 2) = \frac{\pi^3}{6} L(f_5, 1)$$

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$$\begin{aligned} L(f_7, 6) &= \frac{3\pi}{10} L(f_7, 5) = \frac{3\pi^2}{40} L(f_7, 4) = \frac{\pi^3}{80} L(f_7, 3) = \frac{\pi^4}{640} L(f_7, 2) \\ &= \frac{\pi^5}{3840} L(f_7, 1) \end{aligned}$$

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$$\begin{aligned} L(f_9, 8) &= \frac{3\pi}{10} L(f_9, 7) = \frac{3\pi^2}{35} L(f_9, 6) = \frac{4\pi^3}{175} L(f_9, 5) = \frac{\pi^4}{175} L(f_9, 4) \\ &= \frac{\pi^5}{700} L(f_9, 3) = \frac{\pi^6}{2400} L(f_9, 2) = \frac{\pi^7}{5040} L(f_9, 1) \end{aligned}$$

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$$\begin{aligned} L(f_9, 8) &= \frac{3\pi}{10} L(f_9, 7) = \frac{3\pi^2}{35} L(f_9, 6) = \frac{4\pi^3}{175} L(f_9, 5) = \frac{\pi^4}{175} L(f_9, 4) \\ &= \frac{\pi^5}{700} L(f_9, 3) = \frac{\pi^6}{2400} L(f_9, 2) = \frac{\pi^7}{5040} L(f_9, 1) \end{aligned}$$

- By work of Eichler, Shimura and Manin such relations must exist with algebraic numbers.
- $L(f_5, 4) = \frac{\pi^2}{5} L(f_5, 2)$ follows from a result by Fukuhara, Yang ('13).

Q In principle, such evaluations can be rigorously obtained (Rankin's method). Are there implementations?

Conclusions

- Golyshev and Zagier observed that for

$$A(n) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2, \quad f(\tau) = \eta(2\tau)^4 \eta(4\tau)^4 = \sum_{n \geq 1} \alpha_n q^n$$

the known modular congruences have a continuous analog:

weight 4

$$A\left(\frac{p-1}{2}\right) \equiv \alpha_p \pmod{p^2}, \quad A\left(-\frac{1}{2}\right) = \frac{16}{\pi^2} L(f, 2)$$

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Can Zagier's motivic approach (relying on Tate conjecture) be worked out explicitly in these cases?

- Further examples exist. What is the natural framework?

Apéry-like sequences, CM modular forms, hypergeometric series, ...

- How to characterize the analytic interpolations abstractly?

We used suitable binomial sums. But the interpolations are not unique! (Some grow like $\sin(\pi x)$ as $x \rightarrow i\infty$.)

- Polynomial analogs?

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THM

McCarthy,
Osburn,
S 2018

For any odd prime p ,

$$A_\sigma\left(\frac{p-1}{2}\right) \equiv \gamma(p) \pmod{p^2}, \quad \eta^{12}(2\tau) = \sum_{n \geq 1} \gamma(n) q^n \in S_6(\Gamma_0(4))$$

- Here, $A_\sigma(n) = \sum_{\substack{k_1, k_2, k_3, k_4=0 \\ k_1+k_2=k_3+k_4}}^n \prod_{i=1}^4 \binom{n}{k_i} \binom{n+k_i}{k_i}$.

Question:

Zagier-type interpolation?

THANK YOU!

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D. McCarthy, R. Osburn, A. Straub

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Mathematical Proceedings of the Cambridge Philosophical Society, 2018



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R. Osburn, A. Straub, W. Zudilin

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VII

Polynomial analogs

- The natural number n has the q -analog:

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + \dots + q^{n-1}$$

In the limit $q \rightarrow 1$ a q -analog reduces to the classical object.

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In the limit $q \rightarrow 1$ a q -analog reduces to the classical object.

- The q -factorial:

$$[n]_q! = [n]_q [n-1]_q \cdots [1]_q$$

- The q -binomial coefficient:

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} = \binom{n}{n-k}_q$$

EG

$$\binom{6}{2} = \frac{6 \cdot 5}{2} = 3 \cdot 5$$

$$\binom{6}{2}_q = \frac{(1 + q + q^2 + q^3 + q^4)(1 + q + q^2 + q^3 + q^4)}{1 + q}$$

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$$\begin{aligned}\binom{6}{2}_q &= \frac{(1 + q + q^2 + q^3 + q^4)(1 + q + q^2 + q^3 + q^4)}{1 + q} \\ &= (1 - q + q^2) \underbrace{(1 + q + q^2)}_{=[3]_q} \underbrace{(1 + q + q^2 + q^3 + q^4)}_{=[5]_q}\end{aligned}$$

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- The cyclotomic polynomial $\Phi_6(q)$ becomes 1 for $q = 1$ and hence invisible in the classical world

The coefficients of q -binomial coefficients

- Here's some q -binomials in expanded form:

EG

$$\binom{6}{2}_q = q^8 + q^7 + 2q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1$$

$$\begin{aligned} \binom{9}{3}_q &= q^{18} + q^{17} + 2q^{16} + 3q^{15} + 4q^{14} + 5q^{13} + 7q^{12} \\ &\quad + 7q^{11} + 8q^{10} + 8q^9 + 8q^8 + 7q^7 + 7q^6 + 5q^5 \\ &\quad + 4q^4 + 3q^3 + 2q^2 + q + 1 \end{aligned}$$

- The degree of the q -binomial is $k(n - k)$.
- All coefficients are positive!
- In fact, the coefficients are unimodal.

Sylvester, 1878

A few faces of the q -binomial coefficient

The q -binomial coefficient $\binom{n}{k}_q$

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- counts the number of k -dimensional subspaces of \mathbb{F}_q^n .

A q -analog of Babbage's congruence

- Combinatorially, we again obtain:

“ q -Chu-Vandermonde”

$$\binom{2n}{n}_q = \sum_{k=0}^n \binom{n}{k}_q \binom{n}{n-k}_q q^{(n-k)^2}$$

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THM
Clark
1995

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- Note that $\Phi_n(1) = 1$ if n is not a prime power.
- Similar results by Andrews (1999); e.g.:

$$\binom{ap}{bp}_q \equiv q^{(a-b)b\binom{p}{2}} \binom{a}{b}_{q^p} \pmod{[p]_q^2}$$

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- The following answers the question of Andrews to find a q -analog of Wolstenholme's congruence.

THM
S
2011/18

$$\binom{an}{bn}_q \equiv \binom{a}{b}_{q^{n^2}} - (a-b)b \binom{a}{b} \frac{n^2-1}{24} (q^n-1)^2 \pmod{\Phi_n(q)^3}$$

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EG
 $n = 13,$
 $a = 2,$
 $b = 1$

$$\binom{26}{13}_q = 1 + q^{169} - 14(q^{13}-1)^2 + (1+q+\dots+q^{12})^3 f(q)$$

where $f(q) = 14 - 41q + 41q^2 - \dots + q^{132} \in \mathbb{Z}[q]$.

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- Note that $\frac{n^2-1}{24}$ is an integer if $(n, 6) = 1$.
- Ljunggren's classical congruence holds modulo p^{3+r} with r the p -adic valuation of

$$(a-b)ab \binom{a}{b}.$$

Jacobsthal '52

A q -version of the Apéry numbers

- A symmetric q -analog of the Apéry numbers:

$$A_q(n) = \sum_{k=0}^n q^{(n-k)^2} \binom{n}{k}_q^2 \binom{n+k}{k}_q^2$$

This is an explicit form of a q -analog of Krattenthaler, Rivoal and Zudilin (2006).

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- The first few values are:

$$A(0) = 1$$

$$A_q(0) = 1$$

$$A(1) = 5$$

$$A_q(1) = 1 + 3q + q^2$$

$$A(2) = 73$$

$$A_q(2) = 1 + 3q + 9q^2 + 14q^3 + 19q^4 + 14q^5 + 9q^6 + 3q^7 + q^8$$

$$A(3) = 1445$$

$$A_q(3) = 1 + 3q + 9q^2 + 22q^3 + 43q^4 + 76q^5 + 117q^6 + \dots + 3q^{17} + q^{18}$$

THM
S
2014/18

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THM
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THM
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- q -analog and congruences for Almkvist–Zudilin numbers?
(classical supercongruences still open)

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