Interpolated sequences and critical L-values of modular forms

Special Session on Partition Theory and Related Topics
AMS Fall Southeastern Sectional Meeting, Gainesville

Armin Straub

November 3, 2019

University of South Alabama

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \qquad f(\tau) = \eta(2\tau)^4 \eta(4\tau)^4 = \sum_{n \geqslant 1} \alpha_n q^n$$

$$1, 5, 73, 1445, 33001, 819005, 21460825, \ldots$$

$$A(\tfrac{p-1}{2}) \equiv \alpha_p \pmod{p^2}$$

$$A(-\tfrac{1}{2}) = \tfrac{16}{\pi^2} L(f, 2)$$

Joint work with:

Robert Osburn
(University College Dublin)

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2+17n+5)u_n - n^3 u_{n-1}.$$

THM Apéry '78: $\zeta(3) = \sum_{n \geqslant 1} \frac{1}{n^3}$ is irrational.

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2+17n+5)u_n - n^3 u_{n-1}.$$

THM Apéry '78 $\zeta(3) = \sum_{n\geqslant 1} \frac{1}{n^3}$ is irrational.

proof The same recurrence is satisfied by the "near"-integers

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right).$$

Then, $\frac{B(n)}{A(n)} \to \zeta(3)$. But too fast for $\zeta(3)$ to be rational. □

Zagier's search and Apéry-like numbers

- The Apéry numbers $B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}$ for $\zeta(2)$ satisfy

$$(n+1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2 u_{n-1}, \qquad (a,b,c) = (11,3,-1).$$

Q
Beukers
Are there other tuples (a,b,c) for which the solution defined by $u_{-1} = 0$, $u_0 = 1$ is integral?

- The Apéry numbers $B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}$ for $\zeta(2)$ satisfy

$$(n+1)^2 u_{n+1} = (an^2 + an + b) u_n - cn^2 u_{n-1}, \qquad (a,b,c) = (11,3,-1).$$

Q (Beukers): Are there other tuples (a,b,c) for which the solution defined by $u_{-1} = 0$, $u_0 = 1$ is integral?

- Apart from degenerate cases, Zagier found 6 sporadic integer solutions:

*	$C_*(n)$	*	$C_*(n)$
A	$\sum_{k=0}^{n} \binom{n}{k}^3$	**D**	$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{n}$
B	$\sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k 3^{n-3k} \binom{n}{3k} \frac{(3k)!}{k!^3}$	**E**	$\sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \binom{2(n-k)}{n-k}$
C	$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}$	**F**	$\sum_{k=0}^{n} (-1)^k 8^{n-k} \binom{n}{k} C_{\boldsymbol{A}}(k)$

Modularity of Apéry-like numbers

- The Apéry numbers $1, 5, 73, 1145, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n \geqslant 0} A(n) \underbrace{\left(\frac{\eta^{12}(\tau)\eta^{12}(6\tau)}{\eta^{12}(2\tau)\eta^{12}(3\tau)}\right)^n}_{\text{modular function}}.$$

$1 + 5q + 13q^2 + 23q^3 + O(q^4)$ $\qquad$ $q - 12q^2 + 66q^3 + O(q^4)$

Modularity of Apéry-like numbers

- The Apéry numbers $1, 5, 73, 1145, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n\geqslant 0} A(n) \underbrace{\left(\frac{\eta^{12}(\tau)\eta^{12}(6\tau)}{\eta^{12}(2\tau)\eta^{12}(3\tau)}\right)^n}_{\text{modular function}}.$$

$1 + 5q + 13q^2 + 23q^3 + O(q^4)$ $\qquad$ $q - 12q^2 + 66q^3 + O(q^4)$

FACT Not at all evidently, such a **modular parametrization** exists for all known Apéry-like numbers!

- Context:

$f(\tau)$	modular form of weight k
$x(\tau)$	modular function
$y(x)$	such that $y(x(\tau)) = f(\tau)$

Then $y(x)$ satisfies a linear differential equation of order $k+1$.

THM Ahlgren–Ono 2000

For primes $p > 2$, the Apéry numbers for $\zeta(3)$ satisfy

$$A(\tfrac{p-1}{2}) \equiv a_f(p) \pmod{p^2},$$

with $f(\tau) = \eta(2\tau)^4\eta(4\tau)^4 = \sum_{n\geqslant 1} a_f(n)q^n \in S_4(\Gamma_0(8))$.

conjectured (and proved modulo p) by Beukers '87

THM Ahlgren–Ono 2000

For primes $p > 2$, the Apéry numbers for $\zeta(3)$ satisfy

$$A(\tfrac{p-1}{2}) \equiv a_f(p) \qquad (\operatorname{mod} p^2),$$

with $f(\tau) = \eta(2\tau)^4\eta(4\tau)^4 = \sum_{n\geqslant 1} a_f(n)q^n \in S_4(\Gamma_0(8))$.

conjectured (and proved modulo p) by Beukers '87

THM Zagier 2016

$$A(-\tfrac{1}{2}) = \frac{16}{\pi^2}L(f,2)$$

- Here, $A(x) = \sum_{k=0}^{\infty} \binom{x}{k}^2 \binom{x+k}{k}^2$ is absolutely convergent for $x \in \mathbb{C}$.
- Predicted by Golyshev based on motivic considerations, the connection of the Apéry numbers with the double covering of a family of K3 surfaces, and the Tate conjecture.

D. Zagier
Arithmetic and topology of differential equations
Proceedings of the 2016 ECM, 2017

L-value interpolations, cont'd

- Zagier found 6 sporadic integer solutions $C_*(n)$ to: $\qquad$ $*$ one of $\boldsymbol{A}$-$\boldsymbol{F}$

$$(n+1)^2 u_{n+1} = (an^2 + an + b) u_n - cn^2 u_{n-1} \qquad u_{-1} = 0, u_0 = 1$$

THM 1985 - 2019 There exists a weight 3 newform $f_*(\tau) = \sum_{n \geqslant 1} \gamma_{n,*} q^n$, so that

$$C_*\left(\tfrac{p-1}{2}\right) \equiv \gamma_{p,*} \pmod{p}.$$

L-value interpolations, cont'd

- Zagier found 6 sporadic integer solutions $C_*(n)$ to: $\quad * $ one of $\boldsymbol{A}$-$\boldsymbol{F}$

$$(n+1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2 u_{n-1} \qquad u_{-1} = 0, u_0 = 1$$

THM 1985 - 2019 There exists a weight 3 newform $f_*(\tau) = \sum_{n \geqslant 1} \gamma_{n,*} q^n$, so that

$$C_*\left(\tfrac{p-1}{2}\right) \equiv \gamma_{p,*} \pmod{p}.$$

- $\boldsymbol{C}$, $\boldsymbol{D}$ proved by Beukers–Stienstra ('85); $\boldsymbol{A}$ follows from their work
- $\boldsymbol{E}$ proved using a result Verrill ('10); $\boldsymbol{B}$ through p-adic analysis
- $\boldsymbol{F}$ conjectured by Osburn–S and proved by Kazalicki ('19) using Atkin–Swinnerton-Dyer congruences for non-congruence cusp forms

L-value interpolations, cont'd

- Zagier found 6 sporadic integer solutions $C_*(n)$ to: $\quad * $ one of $\boldsymbol{A}$-$\boldsymbol{F}$

$$(n+1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2 u_{n-1} \qquad u_{-1} = 0, u_0 = 1$$

THM 1985 - 2019

There exists a weight 3 newform $f_*(\tau) = \sum_{n \geqslant 1} \gamma_{n,*} q^n$, so that

$$C_*\left(\tfrac{p-1}{2}\right) \equiv \gamma_{p,*} \pmod{p}.$$

- $\boldsymbol{C}$, $\boldsymbol{D}$ proved by Beukers–Stienstra ('85); $\boldsymbol{A}$ follows from their work
- $\boldsymbol{E}$ proved using a result Verrill ('10); $\boldsymbol{B}$ through p-adic analysis
- $\boldsymbol{F}$ conjectured by Osburn–S and proved by Kazalicki ('19) using Atkin–Swinnerton-Dyer congruences for non-congruence cusp forms

THM Osburn S '18

For $*$ one of $\boldsymbol{A}$-$\boldsymbol{F}$, except $\boldsymbol{E}$, there is $\alpha_* \in \mathbb{Z}$ such that

$$C_*(-\tfrac{1}{2}) = \frac{\alpha_*}{\pi^2} L(f_*, 2).$$

- Zagier found 6 sporadic integer solutions $C_*(n)$ to: $\quad * $ one of $\boldsymbol{A}$-$\boldsymbol{F}$

$$(n+1)^2 u_{n+1} = (an^2 + an + b) u_n - cn^2 u_{n-1} \qquad u_{-1} = 0, u_0 = 1$$

THM 1985 - 2019 There exists a weight 3 newform $f_*(\tau) = \sum_{n \geqslant 1} \gamma_{n,*} q^n$, so that

$$C_*\left(\tfrac{p-1}{2}\right) \equiv \gamma_{p,*} \pmod{p}.$$

- $\boldsymbol{C}$, $\boldsymbol{D}$ proved by Beukers–Stienstra ('85); $\boldsymbol{A}$ follows from their work
- $\boldsymbol{E}$ proved using a result Verrill ('10); $\boldsymbol{B}$ through p-adic analysis
- $\boldsymbol{F}$ conjectured by Osburn–S and proved by Kazalicki ('19) using Atkin–Swinnerton-Dyer congruences for non-congruence cusp forms

THM Osburn S '18 For $*$ one of $\boldsymbol{A}$-$\boldsymbol{F}$, except $\boldsymbol{E}$, there is $\alpha_* \in \mathbb{Z}$ such that

$$C_*(-\tfrac{1}{2}) = \frac{\alpha_*}{\pi^2} L(f_*, 2).$$

For sequence $\boldsymbol{E}$, $\quad \operatorname*{res}_{x=-1/2} C_{\boldsymbol{E}}(x) = \frac{6}{\pi^2} L(f_{\boldsymbol{E}}, 1).$

$$C_*(-\tfrac{1}{2}) = \frac{\alpha_*}{\pi^2} L(f_*, 2)$$

*	$C_*(n)$	$f_*(\tau)$	N_*	CM	α_*
A	$\sum_{k=0}^{n} \binom{n}{k}^3$	$\frac{\eta(4\tau)^5\eta(8\tau)^5}{\eta(2\tau)^2\eta(16\tau)^2}$	32	$\mathbb{Q}(\sqrt{-2})$	8
B	$\sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k 3^{n-3k} \binom{n}{3k} \frac{(3k)!}{k!^3}$	$\eta(4\tau)^6$	16	$\mathbb{Q}(\sqrt{-1})$	8
C	$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}$	$\eta(2\tau)^3\eta(6\tau)^3$	12	$\mathbb{Q}(\sqrt{-3})$	12
D	$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{n}$	$\eta(4\tau)^6$	16	$\mathbb{Q}(\sqrt{-1})$	16
E	$\sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \binom{2(n-k)}{n-k}$	$\eta(\tau)^2\eta(2\tau)\eta(4\tau)\eta(8\tau)^2$	8	$\mathbb{Q}(\sqrt{-2})$	6
F	$\sum_{k=0}^{n} (-1)^k 8^{n-k} \binom{n}{k} C_{\boldsymbol{A}}(k)$	$q - 2q^2 + 3q^3 + \dots$	24	$\mathbb{Q}(\sqrt{-6})$	6

Interpolating sequences

Q What is the proper way of defining $C(-\frac{1}{2})$?

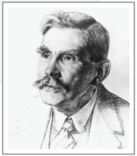
EG $a(n) = n!$ is interpolated by $a(x) = \Gamma(x+1) = \int_0^\infty t^x e^{-t}\, \mathrm{d}t$.

Q What is the proper way of defining $C(-\frac{1}{2})$?

EG $a(n) = n!$ is interpolated by $a(x) = \Gamma(x+1) = \int_0^\infty t^x e^{-t}\,\mathrm{d}t.$

THM Glaisher 1874

$$\int_0^\infty \left(a(0) - a(1)x^2 + a(2)x^4 - \ldots\right)\,\mathrm{d}x = \frac{\pi}{2} a(-\tfrac{1}{2})$$

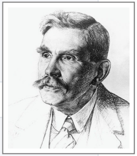


Q What is the proper way of defining $C(-\frac{1}{2})$?

EG $a(n) = n!$ is interpolated by $a(x) = \Gamma(x+1) = \int_0^\infty t^x e^{-t}\,\mathrm{d}t$.

THM Glaisher 1874

$$\int_0^\infty \left(a(0) - a(1)x^2 + a(2)x^4 - \ldots\right)\,\mathrm{d}x = \frac{\pi}{2} a(-\tfrac{1}{2})$$



"poof"

$$\int_0^\infty \frac{1}{1+x^2 S} \cdot a(0)\,\mathrm{d}x = \frac{\pi}{2} S^{-1/2} \cdot a(0)$$

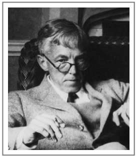
□

(Glaisher's formal proof, simplified by O'Kinealy)

Here, S is the shift operator: $S \cdot b(n) = b(n+1)$

THM Ramanujan Hardy

$$\int_0^\infty x^{s-1}\left(a(0) - x a(1) + x^2 a(2) - \ldots\right)\,\mathrm{d}x = \frac{\pi}{\sin s\pi} a(-s)$$



THM Ramanujan Hardy

$$\int_0^\infty x^{s-1} \left(a(0) - xa(1) + x^2 a(2) - \ldots\right) \mathrm{d}x = \frac{\pi}{\sin s\pi} a(-s)$$

for $0 < \mathrm{Re}\ s < \delta$, provided that

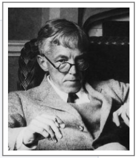
- a is analytic on $H(\delta) = \{z \in \mathbb{C} : \mathrm{Re}\ u \geqslant -\delta\}$,
- $|a(x + iy)| < Ce^{\alpha|x| + \beta|y|}$ for some $\beta < \pi$.

THM Ramanujan Hardy

$$\int_0^\infty x^{s-1}\left(a(0) - xa(1) + x^2 a(2) - \ldots\right) \mathrm{d}x = \frac{\pi}{\sin s\pi} a(-s)$$

for $0 < \mathrm{Re}\ s < \delta$, provided that

- a is analytic on $H(\delta) = \{z \in \mathbb{C} : \mathrm{Re}\ u \geqslant -\delta\}$,
- $|a(x+iy)| < Ce^{\alpha|x|+\beta|y|}$ for some $\beta < \pi$.



COR Carlson 1914

Suppose a satisfies the conditions for RMT. If

$$a(0) = 0, \quad a(1) = 0, \quad a(2) = 0, \quad \ldots,$$

then $a(z) = 0$ identically.

- However, we will see that our interpolations do not arise in this way.

Interpolating sequences

Q What is the proper way of defining $C(-\frac{1}{2})$?

- For Apéry numbers $A(n)$, Zagier used $A(x) = \sum_{k=0}^{\infty} \binom{x}{k}^2 \binom{x+k}{k}^2$.

Q What is the proper way of defining $C(-\frac{1}{2})$?

- For Apéry numbers $A(n)$, Zagier used $A(x) = \sum_{k=0}^{\infty} \binom{x}{k}^2 \binom{x+k}{k}^2$.

EG **Zagier**

$$(x+2)^3 A(x+2) - (2x+3)(17x^2+51x+39)A(x+1)$$
$$+ (x+1)^3 A(x) = 0 \quad \text{for all } x \in \mathbb{Z}_{\geqslant 0}$$

Interpolating sequences

Q What is the proper way of defining $C(-\frac{1}{2})$?

- For Apéry numbers $A(n)$, Zagier used $A(x) = \sum_{k=0}^{\infty} \binom{x}{k}^2 \binom{x+k}{k}^2$.

EG **Zagier**

$$(x+2)^3 A(x+2) - (2x+3)(17x^2+51x+39)A(x+1) + (x+1)^3 A(x) = \frac{8}{\pi^2}(2x+3)\sin^2(\pi x)$$

In particular, $A(x)$ does not satisfy the (vertical) growth conditions of RMT.

Q What is the proper way of defining $C(-\frac{1}{2})$?

- For Apéry numbers $A(n)$, Zagier used $A(x) = \sum_{k=0}^{\infty} \binom{x}{k}^2 \binom{x+k}{k}^2$.

EG **Zagier**

$$(x+2)^3 A(x+2) - (2x+3)(17x^2+51x+39)A(x+1) + (x+1)^3 A(x) = \frac{8}{\pi^2}(2x+3)\sin^2(\pi x)$$

In particular, $A(x)$ does not satisfy the (vertical) growth conditions of RMT.

- For the $\zeta(2)$ Apéry numbers $B(n)$, we use $B(x) = \sum_{k=0}^{\infty} \binom{x}{k}^2 \binom{x+k}{k}$.

  However:
  - The series diverges if $\operatorname{Re} x < -1$.
  - $Q(x, S_x)B(x) = 0$ where $Q(x, S_x)$ is Apéry's recurrence operator.

Q What is the proper way of defining $C(-\frac{1}{2})$?

- For Apéry numbers $A(n)$, Zagier used $A(x) = \sum_{k=0}^{\infty} \binom{x}{k}^2 \binom{x+k}{k}^2$.

diverges for $n \notin \mathbb{Z}_{\geqslant 0}$

EG ($\boldsymbol{C}$)

$$C_{\boldsymbol{C}}(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}$$

Q What is the proper way of defining $C(-\frac{1}{2})$?

- For Apéry numbers $A(n)$, Zagier used $A(x) = \sum_{k=0}^{\infty} \binom{x}{k}^2 \binom{x+k}{k}^2$.

diverges for $n \notin \mathbb{Z}_{\geqslant 0}$

EG **(C)**

$$C_{\boldsymbol{C}}(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} = {}_3F_2\left(\begin{matrix} -n, -n, \frac{1}{2} \\ 1, 1 \end{matrix} \middle| 4\right)$$

Interpolating sequences

Q What is the proper way of defining $C(-\frac{1}{2})$?

- For Apéry numbers $A(n)$, Zagier used $A(x) = \sum_{k=0}^{\infty} \binom{x}{k}^2 \binom{x+k}{k}^2$.

diverges for $n \notin \mathbb{Z}_{\geqslant 0}$

EG (C)

$$C_{\boldsymbol{C}}(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} = {}_3F_2\left(\begin{matrix}-n, -n, \frac{1}{2}\\ 1, 1\end{matrix}\middle| 4\right)$$

We use the interpolation $C_{\boldsymbol{C}}(x) = \operatorname{Re} {}_3F_2\left(\begin{matrix}-x, -x, \frac{1}{2}\\ 1, 1\end{matrix}\middle| 4\right)$.

Q What is the proper way of defining $C(-\frac{1}{2})$?

- For Apéry numbers $A(n)$, Zagier used $A(x) = \sum_{k=0}^{\infty} \binom{x}{k}^2 \binom{x+k}{k}^2$.

diverges for $n \notin \mathbb{Z}_{\geqslant 0}$

EG **($\boldsymbol{C}$)**

$$C_{\boldsymbol{C}}(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} = {}_3F_2\left(\left. \begin{matrix} -n, -n, \frac{1}{2} \\ 1, 1 \end{matrix} \right| 4 \right)$$

We use the interpolation $C_{\boldsymbol{C}}(x) = \operatorname{Re}\, {}_3F_2\left(\left. \begin{matrix} -x, -x, \frac{1}{2} \\ 1, 1 \end{matrix} \right| 4 \right)$.

EG **($\boldsymbol{E}$)**

$$C_{\boldsymbol{E}}(n) = \sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \binom{2(n-k)}{n-k}$$

Q What is the proper way of defining $C(-\frac{1}{2})$?

- For Apéry numbers $A(n)$, Zagier used $A(x) = \sum_{k=0}^{\infty} \binom{x}{k}^2 \binom{x+k}{k}^2$.

diverges for $n \notin \mathbb{Z}_{\geqslant 0}$

EG ($\boldsymbol{C}$)

$$C_{\boldsymbol{C}}(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} = {}_3F_2\left(\left.\begin{matrix} -n, -n, \frac{1}{2} \\ 1, 1 \end{matrix}\right| 4\right)$$

We use the interpolation $C_{\boldsymbol{C}}(x) = \operatorname{Re}\, {}_3F_2\left(\left.\begin{matrix} -x, -x, \frac{1}{2} \\ 1, 1 \end{matrix}\right| 4\right)$.

EG ($\boldsymbol{E}$)

$$C_{\boldsymbol{E}}(n) = \sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \binom{2(n-k)}{n-k} = \binom{2n}{n} {}_3F_2\left(\left.\begin{matrix} -n, -n, \frac{1}{2} \\ \frac{1}{2}-n, 1 \end{matrix}\right| -1\right)$$

This has a simple pole at $n = -\frac{1}{2}$.

Q What is the proper way of defining $C(-\frac{1}{2})$?

- For Apéry numbers $A(n)$, Zagier used $A(x) = \sum_{k=0}^{\infty} \binom{x}{k}^2 \binom{x+k}{k}^2$.

EG

$$C(n) = \sum_{\substack{k_1,k_2,k_3,k_4=0 \\ k_1+k_2=k_3+k_4}}^{n} \prod_{i=1}^{4} \binom{n}{k_i}\binom{n+k_i}{k_i}.$$

How to compute $C(-\frac{1}{2})$?

- RE: order 4, degree 15
- DE: order 7, degree 17
  (2 analytic solutions)

Q What is the proper way of defining $C(-\frac{1}{2})$?

- For Apéry numbers $A(n)$, Zagier used $A(x) = \sum_{k=0}^{\infty} \binom{x}{k}^2 \binom{x+k}{k}^2$.

EG

$$C(n) = \sum_{\substack{k_1,k_2,k_3,k_4=0 \\ k_1+k_2=k_3+k_4}}^{n} \prod_{i=1}^{4} \binom{n}{k_i}\binom{n+k_i}{k_i}.$$

How to compute $C(-\frac{1}{2})$?

- RE: order 4, degree 15
- DE: order 7, degree 17
  (2 analytic solutions)

THM McCarthy, Osburn, S 2018 For any odd prime p,

$$C(\tfrac{p-1}{2}) \equiv \gamma(p) \pmod{p^2}, \qquad \eta^{12}(2\tau) = \sum_{n\geqslant 1} \gamma(n) q^n \in S_6(\Gamma_0(4))$$

Q Is there a Zagier-type interpolation?

Conclusions

- Golyshev and Zagier observed that for

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \qquad f(\tau) = \eta(2\tau)^4 \eta(4\tau)^4 = \sum_{n \geqslant 1} \alpha_n q^n$$

  the known modular congruences have a continuous analog: weight 4

$$A(\tfrac{p-1}{2}) \equiv \alpha_p \pmod{p^2}, \qquad A(-\tfrac{1}{2}) = \tfrac{16}{\pi^2} L(f, 2)$$

Conclusions

- Golyshev and Zagier observed that for

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \qquad f(\tau) = \eta(2\tau)^4 \eta(4\tau)^4 = \sum_{n \geqslant 1} \alpha_n q^n$$

  the known modular congruences have a continuous analog: weight 4

$$A(\tfrac{p-1}{2}) \equiv \alpha_p \pmod{p^2}, \qquad A(-\tfrac{1}{2}) = \tfrac{16}{\pi^2} L(f, 2)$$

- We proved that the same phenomenon holds for:
  - all six sporadic sequences of Zagier weight 3
  - an infinite family of leading coefficients of Brown's cellular integrals

odd weight k

Conclusions

- Golyshev and Zagier observed that for

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \qquad f(\tau) = \eta(2\tau)^4 \eta(4\tau)^4 = \sum_{n \geqslant 1} \alpha_n q^n$$

  the known modular congruences have a continuous analog: weight 4

$$A(\tfrac{p-1}{2}) \equiv \alpha_p \pmod{p^2}, \qquad A(-\tfrac{1}{2}) = \tfrac{16}{\pi^2} L(f, 2)$$

- We proved that the same phenomenon holds for:
  - all six sporadic sequences of Zagier — weight 3
  - an infinite family of leading coefficients of Brown's cellular integrals — odd weight k

- Proofs are computational and not satisfactorily uniform

  Do all of these have the same motivic explanation?

  Can Zagier's motivic approach (relying on Tate conjecture) be worked out explicitly in these cases?

Conclusions

- Golyshev and Zagier observed that for

$$A(n)=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2, \qquad f(\tau)=\eta(2\tau)^4\eta(4\tau)^4=\sum_{n\geqslant 1}\alpha_n q^n$$

  the known modular congruences have a continuous analog: weight 4

$$A(\tfrac{p-1}{2})\equiv\alpha_p \pmod{p^2}, \qquad A(-\tfrac{1}{2})=\tfrac{16}{\pi^2}L(f,2)$$

- We proved that the same phenomenon holds for:
  - all six sporadic sequences of Zagier weight 3
  - an infinite family of leading coefficients of Brown's cellular integrals

  odd weight k

- Proofs are computational and not satisfactorily uniform

  Do all of these have the same motivic explanation?
  Can Zagier's motivic approach (relying on Tate conjecture) be worked out explicitly in these cases?

- Further examples exist. What is the natural framework?

  Apéry-like sequences, CM modular forms, hypergeometric series, ...

- How to characterize the analytic interpolations abstractly?

  We used suitable binomial sums. But the interpolations are not unique! (Some grow like $\sin(\pi x)$ as $x\to i\infty$.)

- Polynomial analogs?

THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`

D. McCarthy, R. Osburn, A. Straub
Sequences, modular forms and cellular integrals
Mathematical Proceedings of the Cambridge Philosophical Society, 2018

R. Osburn, A. Straub
Interpolated sequences and critical L-values of modular forms
Chapter 14 of the book: *Elliptic Integrals, Elliptic Functions and Modular Forms in Quantum Field Theory*
Editors: J. Blümlein, P. Paule and C. Schneider; Springer, 2019, p. 327-349

R. Osburn, A. Straub, W. Zudilin
A modular supercongruence for ${}_6F_5$: An Apéry-like story
Annales de l'Institut Fourier, Vol. 68, Nr. 5, 2018, p. 1987-2004

D. Zagier
Arithmetic and topology of differential equations
Proceedings of the 2016 ECM, 2017

Beukers' proof of the irrationality of $\zeta(3)$

$$I_n = (-1)^n \int_0^1 \int_0^1 \frac{x^n(1-x)^n y^n (1-y)^n}{(1-xy)^{n+1}}\, dxdy$$
$$J_n = \frac{1}{2} \int_0^1 \int_0^1 \int_0^1 \frac{x^n(1-x)^n y^n (1-y)^n w^n (1-w)^n}{(1-(1-xy)w)^{n+1}}\, dxdydw$$

- Beukers showed that

$$I_n = a(n)\zeta(2) + \tilde{a}(n), \qquad J_n = b(n)\zeta(3) + \tilde{b}(n)$$

Beukers' proof of the irrationality of $\zeta(3)$

$$I_n = (-1)^n \int_0^1 \int_0^1 \frac{x^n(1-x)^n y^n (1-y)^n}{(1-xy)^{n+1}} \, dx dy$$

$$J_n = \frac{1}{2} \int_0^1 \int_0^1 \int_0^1 \frac{x^n(1-x)^n y^n (1-y)^n w^n (1-w)^n}{(1-(1-xy)w)^{n+1}} \, dx dy dw$$

- Beukers showed that

$$I_n = a(n)\zeta(2) + \tilde{a}(n), \qquad J_n = b(n)\zeta(3) + \tilde{b}(n)$$

where $\tilde{a}(n), \tilde{b}(n) \in \mathbb{Q}$ and

$$a(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}, \qquad b(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2.$$

$$I_n = (-1)^n \int_0^1 \int_0^1 \frac{x^n(1-x)^n y^n (1-y)^n}{(1-xy)^{n+1}}\, dxdy$$

$$J_n = \frac{1}{2} \int_0^1 \int_0^1 \int_0^1 \frac{x^n(1-x)^n y^n (1-y)^n w^n (1-w)^n}{(1-(1-xy)w)^{n+1}}\, dxdydw$$

- Beukers showed that

$$I_n = a(n)\zeta(2) + \tilde{a}(n), \qquad J_n = b(n)\zeta(3) + \tilde{b}(n)$$

where $\tilde{a}(n), \tilde{b}(n) \in \mathbb{Q}$ and

$$a(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}, \qquad b(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2.$$

- Brown realizes these as period integrals, for $N = 5, 6$, on the moduli space $\mathcal{M}_{0,N}$ of curves of genus 0 with N marked points.

Brown's cellular integrals

THM Brown 2009: Period integrals on $\mathcal{M}_{0,N}$ are $\mathbb{Q}$-linear combinations of multiple zeta values (MZVs). (conjectured by Goncharov–Manin, 2004)

- Examples of such integrals can be written as: $(a_i, b_j, c_{ij} \in \mathbb{Z})$

$$\int_{0<t_1<\ldots<t_{N-3}<1} \prod t_i^{a_i}(1-t_j)^{b_j}(t_i-t_j)^{c_{ij}} dt_1 \ldots dt_{N-3}$$

- Typically involve MZVs of all weights $\leqslant N-3$.

Brown's cellular integrals

THM Brown 2009: Period integrals on $\mathcal{M}_{0,N}$ are $\mathbb{Q}$-linear combinations of multiple zeta values (MZVs). (conjectured by Goncharov–Manin, 2004)

- Examples of such integrals can be written as: $(a_i, b_j, c_{ij} \in \mathbb{Z})$

$$\int_{0<t_1<\ldots<t_{N-3}<1} \prod t_i^{a_i}(1-t_j)^{b_j}(t_i-t_j)^{c_{ij}} dt_1 \ldots dt_{N-3}$$

- Typically involve MZVs of all weights $\leqslant N-3$.
- Brown constructs families of integrals $I_\sigma(n)$, for which MZVs of submaximal weight vanish.
  Here, σ are certain ("convergent") permutations in S_N.

N	5	6	7	8	9	10	11
# of σ	1	1	5	17	105	771	7028

One of Brown's cellular integrals

- One of the 17 permutations for $N = 8$ is $\sigma = (8, 3, 6, 1, 4, 7, 2, 5)$.
- Cellular integral $I_\sigma(n) = \int_\Delta f_\sigma^n \, \omega_\sigma$ where $\Delta : 0 < t_2 < \ldots < t_6 < 1$

$$f_\sigma = \frac{(-t_2)(t_2 - t_3)(t_3 - t_4)(t_4 - t_5)(t_5 - t_6)(t_6 - 1)}{(t_3 - t_6)(t_6)(-t_4)(t_4 - 1)(1 - t_2)(t_2 - t_5)}, \quad \omega_\sigma = \frac{dt_2 dt_3 dt_4 dt_5 dt_6}{(t_3 - t_6)(t_6)(-t_4)(t_4 - 1)(1 - t_2)(t_2 - t_5)}.$$

One of Brown's cellular integrals

- One of the 17 permutations for $N = 8$ is $\sigma = (8, 3, 6, 1, 4, 7, 2, 5)$.
- Cellular integral $I_\sigma(n) = \int_\Delta f_\sigma^n \, \omega_\sigma$ where $\Delta : 0 < t_2 < \ldots < t_6 < 1$

$$f_\sigma = \frac{(-t_2)(t_2 - t_3)(t_3 - t_4)(t_4 - t_5)(t_5 - t_6)(t_6 - 1)}{(t_3 - t_6)(t_6)(-t_4)(t_4 - 1)(1 - t_2)(t_2 - t_5)}, \quad \omega_\sigma = \frac{dt_2 dt_3 dt_4 dt_5 dt_6}{(t_3 - t_6)(t_6)(-t_4)(t_4 - 1)(1 - t_2)(t_2 - t_5)}.$$

EG Panzer: HyperInt

$$I_\sigma(0) = 16\zeta(5) - 8\zeta(3)\zeta(2)$$
$$I_\sigma(1) = 33 I_\sigma(0) - 432\zeta(3) + 316\zeta(2) - 26$$
$$I_\sigma(2) = 8929 I_\sigma(0) - 117500\zeta(3) + \frac{515189}{6}\zeta(2) - \frac{331063}{48}$$

One of Brown's cellular integrals

- One of the 17 permutations for $N = 8$ is $\sigma = (8, 3, 6, 1, 4, 7, 2, 5)$.
- Cellular integral $I_\sigma(n) = \int_\Delta f_\sigma^n \, \omega_\sigma$ where $\Delta : 0 < t_2 < \ldots < t_6 < 1$

$$f_\sigma = \frac{(-t_2)(t_2 - t_3)(t_3 - t_4)(t_4 - t_5)(t_5 - t_6)(t_6 - 1)}{(t_3 - t_6)(t_6)(-t_4)(t_4 - 1)(1 - t_2)(t_2 - t_5)}, \quad \omega_\sigma = \frac{dt_2 dt_3 dt_4 dt_5 dt_6}{(t_3 - t_6)(t_6)(-t_4)(t_4 - 1)(1 - t_2)(t_2 - t_5)}.$$

EG
Panzer: HyperInt

$$I_\sigma(0) = 16\zeta(5) - 8\zeta(3)\zeta(2)$$
$$I_\sigma(1) = 33 I_\sigma(0) - 432\zeta(3) + 316\zeta(2) - 26$$
$$I_\sigma(2) = 8929 I_\sigma(0) - 117500\zeta(3) + \tfrac{515189}{6}\zeta(2) - \tfrac{331063}{48}$$

- OGF of $I_\sigma(n)$ satisfies a Picard–Fuchs DE of order 7 (Lairez).
  With 2-dimensional space of analytic solutions at 0.

One of Brown's cellular integrals

- One of the 17 permutations for $N = 8$ is $\sigma = (8, 3, 6, 1, 4, 7, 2, 5)$.
- Cellular integral $I_\sigma(n) = \int_\Delta f_\sigma^n \, \omega_\sigma$ where $\quad \Delta : 0 < t_2 < \ldots < t_6 < 1$

$$f_\sigma = \frac{(-t_2)(t_2-t_3)(t_3-t_4)(t_4-t_5)(t_5-t_6)(t_6-1)}{(t_3-t_6)(t_6)(-t_4)(t_4-1)(1-t_2)(t_2-t_5)}, \quad \omega_\sigma = \frac{dt_2 dt_3 dt_4 dt_5 dt_6}{(t_3-t_6)(t_6)(-t_4)(t_4-1)(1-t_2)(t_2-t_5)}.$$

EG
Panzer: HyperInt

$$I_\sigma(0) = 16\zeta(5) - 8\zeta(3)\zeta(2)$$
$$I_\sigma(1) = 33 I_\sigma(0) - 432\zeta(3) + 316\zeta(2) - 26$$
$$I_\sigma(2) = 8929 I_\sigma(0) - 117500\zeta(3) + \frac{515189}{6}\zeta(2) - \frac{331063}{48}$$

- OGF of $I_\sigma(n)$ satisfies a Picard–Fuchs DE of order 7 (Lairez).
  With 2-dimensional space of analytic solutions at 0.
- The leading coefficients of $I_\sigma(n)$ are:

$$1, 33, 8929, 4124193, 2435948001, 1657775448033, \ldots$$

One of Brown's cellular integrals, cont'd

- One of the 17 permutations for $N = 8$ is $\sigma = (8, 3, 6, 1, 4, 7, 2, 5)$.
- Cellular integral $I_\sigma(n) = \int_\Delta f_\sigma^n \, \omega_\sigma$ where
- The leading coefficients $A_\sigma(n)$ of $I_\sigma(n)$ are:

$$1, 33, 8929, 4124193, 2435948001, 1657775448033, \ldots$$

LEM McCarthy, Osburn, S 2018

$$A_\sigma(n) = \sum_{\substack{k_1,k_2,k_3,k_4=0 \\ k_1+k_2=k_3+k_4}}^{n} \prod_{i=1}^{4} \binom{n}{k_i}\binom{n+k_i}{k_i}$$

One of Brown's cellular integrals, cont'd

- One of the 17 permutations for $N = 8$ is $\sigma = (8, 3, 6, 1, 4, 7, 2, 5)$.
- Cellular integral $I_\sigma(n) = \int_\Delta f_\sigma^n \, \omega_\sigma$ where
- The leading coefficients $A_\sigma(n)$ of $I_\sigma(n)$ are:

$$1, 33, 8929, 4124193, 2435948001, 1657775448033, \ldots$$

LEM McCarthy, Osburn, S 2018

$$A_\sigma(n) = \sum_{\substack{k_1,k_2,k_3,k_4=0 \\ k_1+k_2=k_3+k_4}}^{n} \prod_{i=1}^{4} \binom{n}{k_i}\binom{n+k_i}{k_i}$$

CONJ McCarthy, Osburn, S 2018 For each $N \geqslant 5$ and convergent σ_N, the leading coefficients $A_{\sigma_N}(n)$ satisfy $(p \geqslant 5)$

$$A_{\sigma_N}(mp^r) \equiv A_{\sigma_N}(mp^{r-1}) \pmod{p^{3r}}.$$

For $N = 5, 6$ these are the supercongruences proved by Beukers and Coster.

One of Brown's cellular integrals, cont'd

- One of the 17 permutations for $N = 8$ is $\sigma = (8, 3, 6, 1, 4, 7, 2, 5)$.
- Cellular integral $I_\sigma(n) = \int_\Delta f_\sigma^n \, \omega_\sigma$ where
- The leading coefficients $A_\sigma(n)$ of $I_\sigma(n)$ are:

$$1, 33, 8929, 4124193, 2435948001, 1657775448033, \ldots$$

LEM McCarthy, Osburn, S 2018

$$A_\sigma(n) = \sum_{\substack{k_1,k_2,k_3,k_4=0\\k_1+k_2=k_3+k_4}}^{n} \prod_{i=1}^{4} \binom{n}{k_i}\binom{n+k_i}{k_i}$$

THM McCarthy, Osburn, S 2018

For any odd prime p,

$$A_\sigma\left(\frac{p-1}{2}\right) \equiv \gamma(p) \pmod{p^2}.$$

where $\eta^{12}(2\tau) = \sum_{n \geqslant 1} \gamma(n) q^n$ is the unique newform in $S_6(\Gamma_0(4))$.

THM Ahlgren–Ono '00

For any odd prime p, the Apéry numbers for $\zeta(3)$ satisfy

$$A\left(\frac{p-1}{2}\right) \equiv \alpha(p) \pmod{p^2},$$

with $\eta(2\tau)^4\eta(4\tau)^4 = \sum_{n\geqslant 1} \alpha(n) q^n$ the unique newform in $S_4(\Gamma_0(8))$.

THM Ahlgren '01

For any prime $p \geqslant 5$, the Apéry numbers for $\zeta(2)$ satisfy

$$B\left(\frac{p-1}{2}\right) \equiv \beta(p) \pmod{p^2},$$

with $\eta(4\tau)^6 = \sum_{n\geqslant 1} \beta(n) q^n$ the unique newform in $S_3(\Gamma_0(16), (\frac{-4}{\cdot}))$.

- conjectured (and proved modulo p) by Beukers '87

Congruences and interpolations for cellular integrals

- For an explicit family σ_N of convergent configurations, $A_{\sigma_N}(n) = C_{\boldsymbol{D}}(n)^{(N-3)/2}$.

Congruences and interpolations for cellular integrals

- For an explicit family σ_N of convergent configurations, $A_{\sigma_N}(n) = C_{\boldsymbol{D}}(n)^{(N-3)/2}$.
- For odd $k \geqslant 3$, consider the weight k binary theta series

$$f_k(\tau) = \frac{1}{4} \sum_{(n,m) \in \mathbb{Z}^2} (-1)^{m(k-1)/2} (n - im)^{k-1} q^{n^2+m^2} =: \sum_{n \geqslant 1} \gamma_k(n) q^n.$$

Congruences and interpolations for cellular integrals

- For an explicit family σ_N of convergent configurations, $A_{\sigma_N}(n) = C_{\boldsymbol{D}}(n)^{(N-3)/2}$.
- For odd $k \geqslant 3$, consider the weight k binary theta series

$$f_k(\tau) = \frac{1}{4} \sum_{(n,m)\in\mathbb{Z}^2} (-1)^{m(k-1)/2}(n-im)^{k-1} q^{n^2+m^2} =: \sum_{n\geqslant 1} \gamma_k(n) q^n.$$

THM McCarthy, OS '18: Let $N \geqslant 5$ be odd and $k = N-2$. Then, for all primes $p \geqslant 5$,

$$A_{\sigma_N}(\tfrac{p-1}{2}) \equiv \gamma_k(p) \pmod{p^2}.$$

Congruences and interpolations for cellular integrals

- For an explicit family σ_N of convergent configurations, $A_{\sigma_N}(n) = C_{\boldsymbol{D}}(n)^{(N-3)/2}$.
- For odd $k \geqslant 3$, consider the weight k binary theta series

$$f_k(\tau) = \frac{1}{4} \sum_{(n,m)\in\mathbb{Z}^2} (-1)^{m(k-1)/2}(n-im)^{k-1}q^{n^2+m^2} =: \sum_{n\geqslant 1} \gamma_k(n) q^n.$$

THM McCarthy, OS '18 — Let $N \geqslant 5$ be odd and $k = N-2$. Then, for all primes $p \geqslant 5$,

$$A_{\sigma_N}\left(\tfrac{p-1}{2}\right) \equiv \gamma_k(p) \pmod{p^2}.$$

THM OS '18 — Let $N \geqslant 5$ be odd and $k = N-2$. Then,

$$A_{\sigma_N}\left(-\tfrac{1}{2}\right) = \frac{\alpha_k}{\pi^{k-1}} L(f_k, k-1),$$

where α_k are explicit rational numbers, defined recursively.

THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`

D. McCarthy, R. Osburn, A. Straub
Sequences, modular forms and cellular integrals
Mathematical Proceedings of the Cambridge Philosophical Society, 2018

R. Osburn, A. Straub
Interpolated sequences and critical L-values of modular forms
Chapter 14 of the book: *Elliptic Integrals, Elliptic Functions and Modular Forms in Quantum Field Theory*
Editors: J. Blümlein, P. Paule and C. Schneider; Springer, 2019, p. 327-349

R. Osburn, A. Straub, W. Zudilin
A modular supercongruence for ${}_6F_5$: An Apéry-like story
Annales de l'Institut Fourier, Vol. 68, Nr. 5, 2018, p. 1987-2004

D. Zagier
Arithmetic and topology of differential equations
Proceedings of the 2016 ECM, 2017

THANK YOU!

Slides for this talk will be available from my website:
http://arminstraub.com/talks

D. McCarthy, R. Osburn, A. Straub
Sequences, modular forms and cellular integrals
Mathematical Proceedings of the Cambridge Philosophical Society, 2018

R. Osburn, A. Straub
Interpolated sequences and critical L-values of modular forms
Chapter 14 of the book: *Elliptic Integrals, Elliptic Functions and Modular Forms in Quantum Field Theory*
Editors: J. Blümlein, P. Paule and C. Schneider; Springer, 2019, p. 327-349

R. Osburn, A. Straub, W. Zudilin
A modular supercongruence for ${}_6F_5$: An Apéry-like story
Annales de l'Institut Fourier, Vol. 68, Nr. 5, 2018, p. 1987-2004

D. Zagier
Arithmetic and topology of differential equations
Proceedings of the 2016 ECM, 2017