Gauss congruences

ICOMAS 2018: Special Session on Analytic Number Theory
The University of Memphis

Armin Straub

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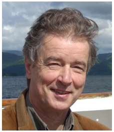
University of South Alabama

based on joint work with  **and** 

Frits Beukers
(Utrecht University)

Marc Houben
(Utrecht University)

- Given a series

$$F(x_1, \ldots, x_d) = \sum_{n_1, \ldots, n_d \geqslant 0} a(n_1, \ldots, n_d) x_1^{n_1} \cdots x_d^{n_d},$$

its **diagonal coefficients** are the coefficients $a(n, \ldots, n)$.

EG The diagonal coefficients of

$$\frac{1}{1 - x - y} = \sum_{n=0}^{\infty} (x + y)^n$$

are the central binomial coefficients $\binom{2n}{n}$.

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are the central binomial coefficients $\binom{2n}{n}$.
For comparison, their univariate generating function is

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1 - 4x}}.$$

EG The **Lucas numbers** L_n have GF $\frac{2-x}{1-x-x^2}$.

$L_{n+1} = L_n + L_{n-1}$
$L_0 = 2, L_1 = 1$

- The sequences with rational GF are precisely the C**-finite** ones.

Introduction: Rational generating functions

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EG The **Delannoy numbers** have GF $\frac{1}{\sqrt{1-6x+x^2}}$.

$D_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}$

They are the diagonal of $\frac{1}{1-x-y-xy}$.

- The sequences with algebraic GF are precisely the diagonals of 2-variable rational functions.

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THM Gessel, Zeilberger, Lipshitz 1981–88

The diagonal of a rational function is D-finite.

More generally, the diagonal of a D-finite function is D-finite.

$F \in K[[x_1, \ldots, x_d]]$ is D-finite if its partial derivatives span a finite-dimensional vector space over $K(x_1, \ldots, x_d)$.

EG

The **Franel numbers** $\sum_{k=0}^{n} \binom{n}{k}^3$ are the diagonal of

$$\frac{1}{1-x-y-z+4xyz}.$$

Their GF is

$$\frac{1}{1-2x}\,{}_2F_1\left(\begin{matrix}\frac{1}{3},\frac{2}{3}\\ 1\end{matrix}\middle|\frac{27x^2}{(1-2x)^3}\right).$$

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- Not at all unique! The Franel numbers are also the diagonal of

$$\frac{1}{(1-x)(1-y)(1-z)-xyz}.$$

THM
S 2014

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- Univariate generating function:

$$\sum_{n\geqslant 0} A(n)x^n = \frac{17-x-z}{4\sqrt{2}(1+x+z)^{3/2}}\, {}_3F_2\left(\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ 1,1\end{matrix}\middle| -\frac{1024x}{(1-x+z)^4}\right),$$

where $z=\sqrt{1-34x+x^2}$.

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- Well-developed theory of multivariate asymptotics — e.g., Pemantle–Wilson
- Such diagonals are algebraic modulo p^r. — Furstenberg, Deligne '67, '84

  Automatically leads to congruences such as

$$A(n) \equiv \begin{cases} 1 & \pmod 8, \quad \text{if } n \text{ even},\\ 5 & \pmod 8, \quad \text{if } n \text{ odd}.\end{cases}$$

Chowla–Cowles–Cowles '80
Rowland–Yassawi '13

The classical Gauss congruence

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$$\sum_{d|m} \mu(\tfrac{m}{d}) a^d \equiv 0 \pmod{m}$$

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EG If $m = p^r$ then only $d = p^r$, $d = p^{r-1}$ contribute, and we get

$$a^{p^r} \equiv a^{p^{r-1}} \pmod{p^r}.$$

DEF $a(n)$ satisfies the **Gauss congruences** if, for all primes p,

$$a(mp^r) \equiv a(mp^{r-1}) \pmod{p^r}.$$

Equivalently, $\sum_{d|m} \mu(\frac{m}{d}) a(d) \equiv 0 \pmod{m}$.

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- $a(n) = D_n$ **Delannoy numbers**: $D_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k}$

- Later, we allow $a(n) \in \mathbb{Q}$. If the Gauss congruences hold for all but finitely many p, we say that the sequence (or its GF) has the **Gauss property**.
- Similarly, for multivariate sequences $a(\boldsymbol{n})$, we require

$$a(\boldsymbol{m}p^r) \equiv a(\boldsymbol{m}p^{r-1}) \pmod{p^r}.$$

More sequences satisfying Gauss congruences

$$a(mp^r) \equiv a(mp^{r-1}) \pmod{p^r} \tag{G}$$

- **realizable** sequences $a(n)$, i.e., for some map $T : X \to X$,

$$a(n) = \#\{x \in X : T^n x = x\} \qquad \text{"points of period } n\text{"}$$

Everest–van der Poorten–Puri–Ward '02, Arias de Reyna '05

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- (G) is equivalent to $\exp\left(\sum_{n=1}^{\infty} \frac{a(n)}{n} T^n\right) \in \mathbb{Z}[[T]]$.

  This is a natural condition in **formal group theory**.

Minton's theorem

THM (Minton, 2014) $f \in \mathbb{Q}(x)$ has the Gauss property if and only if f is a $\mathbb{Q}$-linear combination of functions $xu'(x)/u(x)$, with $u \in \mathbb{Z}[x]$.

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- If $u(x) = \prod_{i=1}^{s}(1 - \alpha_i x)$ then

$$x\frac{u'(x)}{u(x)} = -\sum_{i=1}^{s} \frac{\alpha_i x}{1 - \alpha_i x} = s - \sum_{i=1}^{s} \frac{1}{1 - \alpha_i x}.$$

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- Assuming the α_i are distinct,

$$\sum_{i=1}^{s} \frac{1}{1 - \alpha_i x} = \sum_{n \geqslant 0} \left(\sum_{i=1}^{s} \alpha_i^n \right) x^n = \sum_{n \geqslant 0} \mathrm{trace}(M^n) x^n,$$

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  where M is the companion matrix of $\prod_{i=1}^{s}(x - \alpha_i) = x^s u(1/x)$.
- Minton: No new C-finite sequences with the Gauss property!
- Can we generalize from C-finite towards D-finite?

THM Beukers, Houben, S 2017

Let $f_1, \ldots, f_m \in \mathbb{Q}(\boldsymbol{x}) = \mathbb{Q}(x_1, \ldots, x_n)$ be nonzero. Then

$$\frac{x_1 \cdots x_m}{f_1 \cdots f_m} \det \left(\frac{\partial f_j}{\partial x_i} \right)_{i,j=1,\ldots,m} \tag{D}$$

has the Gauss property.

Interesting detail: true for any of the different Laurent expansions of multivariate rational functions

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EG Consider $Q = 1 - x - y - z + 4xyz$:

$$f_1 = Q \quad \Longrightarrow \quad \text{(D)} = \frac{-x + 4xyz}{Q}$$

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In particular, $\dfrac{1}{1 - x - y - z + 4xyz}$ has the Gauss property.

There is nothing special about 4 in this argument.

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THM BHS

Let $P, Q \in \mathbb{Z}[\boldsymbol{x}]$ with Q is linear in each variable.

Then P/Q has the Gauss property if and only if $N(P) \subseteq N(Q)$.

- Here, $N(Q)$ is the Newton polytope of Q.
- In this case, $N(Q) = \operatorname{supp}(Q) \subseteq \{0,1\}^n$.

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PROP BHS

Let $P, Q \in \mathbb{Z}[\boldsymbol{x}^{\pm 1}]$.
If P/Q has the Gauss property, then $N(P) \subseteq N(Q)$.

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- Yes, for $f = P/Q$ with Q in two variables and total degree 2.

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EG

Can $\dfrac{x(x + y + y^2 + 2xy^2)}{1 + 3x + 3y + 2x^2 + 2y^2 + xy - 2x^2y^2}$ be written in that form?

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EG Beukers, Houben, S 2017

The **Delannoy numbers** D_{n_1,n_2} are characterized by

$$\frac{1}{1-x-y-xy} = \sum_{n_1,n_2=0}^{\infty} D_{n_1,n_2} x^{n_1} y^{n_2}.$$

Application: Delannoy numbers

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By the theorem, the following have the Gauss property:

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In other words, for $\boldsymbol{\delta} \in \{0,1\}^2$,

$$D_{\boldsymbol{m}p^r-\boldsymbol{\delta}} \equiv D_{\boldsymbol{m}p^{r-1}-\boldsymbol{\delta}} \pmod{p^r}.$$

Some open problems

- Which rational functions have the **Gauss property**?

$$A(\boldsymbol{n}p^r) \equiv A(\boldsymbol{n}p^{r-1}) \pmod{p^r}$$

  When are these necessarily combinations of $\frac{x_1 \cdots x_m}{f_1 \cdots f_m} \det \left(\frac{\partial f_j}{\partial x_i} \right)$?

- Which rational functions satisfy **supercongruences**?

$$A(\boldsymbol{n}p^r) \equiv A(\boldsymbol{n}p^{r-1}) \pmod{p^{kr}}, \quad k > 1$$

  And can we prove these?

$$\frac{1}{1-(x+y+z)+4xyz}, \quad \frac{1}{1-(x+y+z+w)+27xyzw}$$

- Is there a rational function in three variables with the $\zeta(3)$-Apéry numbers as diagonal?

THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`

F. Beukers, M. Houben, A. Straub
Gauss congruences for rational functions in several variables
Preprint, 2017. arXiv:1710.00423

A. Straub
Multivariate Apéry numbers and supercongruences of rational functions
Algebra & Number Theory, Vol. 8, Nr. 8, 2014, p. 1985-2008

Bonus

Apéry-like sequences

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

  $$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

  satisfy

  $$(n+1)^3 A(n+1) = (2n+1)(17n^2+17n+5)A(n) - n^3 A(n-1).$$

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THM Apéry '78: $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.

proof The same recurrence is satisfied by the "near"-integers

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right).$$

Then, $\frac{B(n)}{A(n)} \to \zeta(3)$. But too fast for $\zeta(3)$ to be rational. □

- Recurrence for Apéry numbers is the case $(a, b, c) = (17, 5, 1)$ of

$$(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - cn^3 u_{n-1}.$$

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Q **Beukers, Zagier** Are there other tuples (a,b,c) for which the solution defined by $u_{-1} = 0$, $u_0 = 1$ is integral?

- Essentially, only 14 tuples (a,b,c) found. (Almkvist–Zudilin)
  - 4 hypergeometric and 4 Legendrian solutions (with generating functions

$$ {}_3F_2\left(\left. \begin{matrix} \frac{1}{2}, \alpha, 1-\alpha \\ 1, 1 \end{matrix} \right| 4C_\alpha z \right), \qquad \frac{1}{1-C_\alpha z} {}_2F_1\left(\left. \begin{matrix} \alpha, 1-\alpha \\ 1 \end{matrix} \right| \frac{-C_\alpha z}{1-C_\alpha z} \right)^2, $$

    with $\alpha = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$ and $C_\alpha = 2^4, 3^3, 2^6, 2^4 \cdot 3^3$)
  - 6 sporadic solutions

- Similar (and intertwined) story for:
  - $(n+1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2 u_{n-1}$ (Beukers, Zagier)
  - $(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - n(cn^2 + d)u_{n-1}$ (Cooper)

The six sporadic Apéry-like numbers

(a,b,c)	$A(n)$	
$(17,5,1)$	$\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$	Apéry numbers
$(12,4,16)$	$\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$	
$(10,4,64)$	$\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$	Domb numbers
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$	Almkvist–Zudilin numbers
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \binom{4n-5k}{3n}$	
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$	

Supercongruences for Apéry numbers

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THM Beukers, Coster '85, '88

The Apéry numbers satisfy the **supercongruence** $(p \geqslant 5)$

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EG For primes p, simple combinatorics proves the congruence
$$\binom{2p}{p} = \sum_k \binom{p}{k}\binom{p}{p-k} \equiv 1 + 1 \pmod{p^2}.$$

For $p \geqslant 5$, Wolstenholme's congruence shows that, in fact,
$$\binom{2p}{p} \equiv 2 \pmod{p^3}.$$

Supercongruences for Apéry-like numbers

Robert Osburn
(University of Dublin)

Brundaban Sahu
(NISER, India)

- Conjecturally, supercongruences like

$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}$$

hold for all Apéry-like numbers.

Osburn–Sahu '09

- Current state of affairs for the six sporadic sequences from earlier:

(a,b,c)	$A(n)$	
$(17,5,1)$	$\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$	Beukers, Coster '87-'88
$(12,4,16)$	$\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$	Osburn–Sahu–S '16
$(10,4,64)$	$\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$	Osburn–Sahu '11
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$	open; modulo p^3 Amdeberhan–Tauraso '16
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \binom{4n-5k}{3n}$	Osburn–Sahu–S '16
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$	Gorodetsky '18

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Define $A(\boldsymbol{n}) = A(n_1, n_2, n_3, n_4)$ by

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4} = \sum_{\boldsymbol{n} \in \mathbb{Z}_{\geqslant 0}^4} A(\boldsymbol{n}) \boldsymbol{x}^{\boldsymbol{n}}.$$

- The Apéry numbers are the diagonal coefficients.
- For $p \geqslant 5$, we have the **multivariate supercongruences**

$$A(\boldsymbol{n} p^r) \equiv A(\boldsymbol{n} p^{r-1}) \pmod{p^{3r}}.$$

THM S 2014

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- $\sum_{n\geqslant 0} a(n)x^n = F(x) \quad \Longrightarrow \quad \sum_{n\geqslant 0} a(pn)x^{pn} = \frac{1}{p}\sum_{k=0}^{p-1} F(\zeta_p^k x)$ $\qquad \zeta_p = e^{2\pi i/p}$
- Hence, both $A(\boldsymbol{n}p^r)$ and $A(\boldsymbol{n}p^{r-1})$ have rational generating function.

  The proof, however, relies on an explicit binomial sum for the coefficients.

Multivariate supercongruences

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- By MacMahon's Master Theorem,

$$A(\boldsymbol{n}) = \sum_{k \in \mathbb{Z}} \binom{n_1}{k} \binom{n_3}{k} \binom{n_1 + n_2 - k}{n_1} \binom{n_3 + n_4 - k}{n_3}.$$

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- Because $A(n-1) = A(-n,-n,-n,-n)$, we also find

$$A(mp^r - 1) \equiv A(mp^{r-1} - 1) \pmod{p^{3r}}.$$

Beukers '85

An infinite family of rational functions

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Let $\lambda \in \mathbb{Z}_{>0}^{\ell}$ with $d = \lambda_1 + \ldots + \lambda_\ell$. Define $A_\lambda(\boldsymbol{n})$ by

$$\frac{1}{\prod_{1 \leqslant j \leqslant \ell} \left[1 - \sum_{1 \leqslant r \leqslant \lambda_j} x_{\lambda_1 + \ldots + \lambda_{j-1} + r}\right] - x_1 x_2 \cdots x_d} = \sum_{\boldsymbol{n} \in \mathbb{Z}_{\geqslant 0}^d} A_\lambda(\boldsymbol{n}) \boldsymbol{x}^{\boldsymbol{n}}.$$

- If $\ell \geqslant 2$, then, for all primes p,

$$A_\lambda(\boldsymbol{n} p^r) \equiv A_\lambda(\boldsymbol{n} p^{r-1}) \pmod{p^{2r}}.$$

- If $\ell \geqslant 2$ and $\max(\lambda_1, \ldots, \lambda_\ell) \leqslant 2$, then, for primes $p \geqslant 5$,

$$A_\lambda(\boldsymbol{n} p^r) \equiv A_\lambda(\boldsymbol{n} p^{r-1}) \pmod{p^{3r}}.$$

EG

$\lambda = (2, 2)$

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4}$$

$\lambda = (2, 1)$

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3) - x_1 x_2 x_3}$$

Further examples

EG

$$\frac{1}{(1-x_1-x_2)(1-x_3)-x_1x_2x_3}$$

has as diagonal the Apéry-like numbers, associated with $\zeta(2)$,

$$B(n)=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}.$$

EG

$$\frac{1}{(1-x_1)(1-x_2)\cdots(1-x_d)-x_1x_2\cdots x_d}$$

has as diagonal the numbers

$d=3$: Franel, $d=4$: Yang–Zudilin

$$Y_d(n)=\sum_{k=0}^{n}\binom{n}{k}^d.$$

- In each case, we obtain supercongruences generalizing results of Coster (1988) and Chan–Cooper–Sica (2010).

CONJ
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The coefficients $Z(\boldsymbol{n})$ of

$$\frac{1}{1-(x_1+x_2+x_3+x_4)+27x_1x_2x_3x_4} = \sum_{\boldsymbol{n}\in\mathbb{Z}_{\geqslant 0}^4} Z(\boldsymbol{n})\boldsymbol{x}^{\boldsymbol{n}}$$

satisfy, for $p \geqslant 5$, the multivariate supercongruences

$$Z(\boldsymbol{n}p^r) \equiv Z(\boldsymbol{n}p^{r-1}) \pmod{p^{3r}}.$$

- Here, the diagonal coefficients are the **Almkvist–Zudilin numbers**

$$Z(n) = \sum_{k=0}^{n} (-3)^{n-3k} \binom{n}{3k}\binom{n+k}{n}\frac{(3k)!}{k!^3},$$

for which the univariate congruences are still open.

THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`

F. Beukers, M. Houben, A. Straub
Gauss congruences for rational functions in several variables
Preprint, 2017. arXiv:1710.00423

A. Straub
Multivariate Apéry numbers and supercongruences of rational functions
Algebra & Number Theory, Vol. 8, Nr. 8, 2014, p. 1985-2008