A gumbo with hints of partitions, modular forms, special integer sequences and supercongruences

Number Theory Seminar
University of Illinois at Urbana-Champaign

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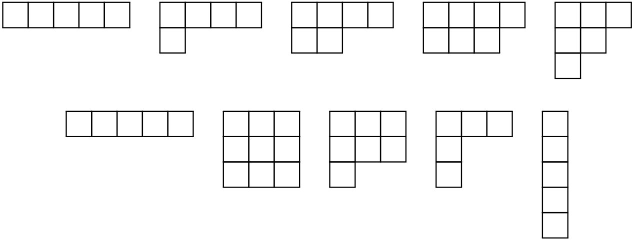
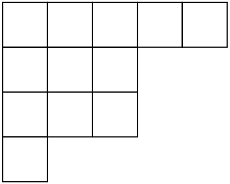
Mar 16, 2017

University of South Alabama



Core partitions

- The integer partition $(5, 3, 3, 1)$ has Young diagram:



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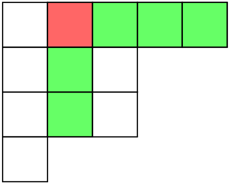

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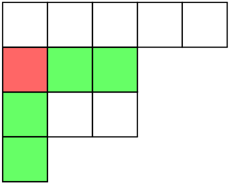

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LEM If a partition is t-core, then it is also rt-core for $r = 1, 2, 3 \ldots$

The number of core partitions

- Using the theory of modular forms, Granville and Ono (1996) showed:

(The case $t = p$ of this completed the classification of simple groups with defect zero Brauer p-blocks.)

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- If $c_t(n)$ is the number of t-core partitions of n, then

$$\sum_{n=0}^{\infty} c_t(n) q^n = \prod_{n=1}^{\infty} \frac{(1-q^{tn})^t}{1-q^n}.$$

$$\sum_{n=0}^{\infty} c_2(n) q^n = \sum_{n=0}^{\infty} q^{\frac{1}{2}n(n+1)}, \quad \sum_{n=0}^{\infty} c_3(n) q^n = 1 + q + 2q^2 + 2q^4 + q^5 + 2q^6 + q^8 + \ldots$$

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COR The total number of t-core partitions is infinite.

Though this is probably the most complicated way possible to see that...

Counting core partitions

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- Ford, Mai and Sze (2009) show that the number of self-conjugate (s,t)-core partitions is

$$\binom{\lfloor s/2 \rfloor + \lfloor t/2 \rfloor}{\lfloor s/2 \rfloor}.$$

Core partitions into distinct parts

- Amdeberhan raises the interesting problem of counting the number of special partitions which are t-core for certain values of t.

CONJ The number of $(s, s+1)$-core partitions into distinct parts equals the Fibonacci number F_{s+1}.

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- He further conjectured that the largest possible size of an $(s, s+1)$-core partition into distinct parts is $\lfloor s(s+1)/6 \rfloor$, and that there is a unique such largest partition unless $s \equiv 1$ modulo 3, in which case there are two partitions of maximum size.
- Amdeberhan also conjectured that the total size of these partitions is

$$\sum_{i+j+k=s+1} F_i F_j F_k.$$

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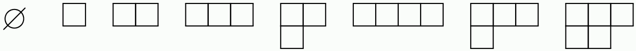
EG $s=5$, $F_6=8$



A two-parameter generalization

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Let $N_d(s)$ be the number of $(s, ds-1)$-core partitions into distinct parts. Then, $N_d(1) = 1$, $N_d(2) = d$ and

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EG

The first few generalized Fibonacci polynomials $N_d(s)$ are

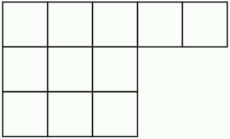
$$1, \quad d, \quad 2d, \quad d(d+2), \quad d(3d+2), \quad d(d^2+5d+2), \ldots$$

For $d = 1$, we recover the usual Fibonacci numbers.
For $d = 2$, we find $N_2(s) = 2^{s-1}$.

- Nice proof (and more!) via abaci structures by Nath and Sellers (2016).

The perimeter of a partition

DEF The **perimeter** of a partition is the maximum hook length in λ.

EG The partition  has perimeter 7.

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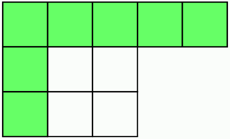
EG The partition  has perimeter 7.

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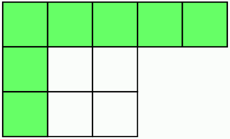

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- Introduced (up to a shift by 1) by Corteel and Lovejoy (2004) in their study of overpartitions.
- The perimeter is the largest part plus the number of parts (minus 1).
- The **rank** is the largest part minus the number of parts.

Euler's theorem and a simple analog

THM Euler

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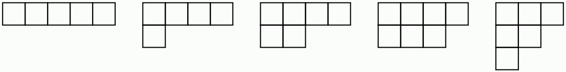

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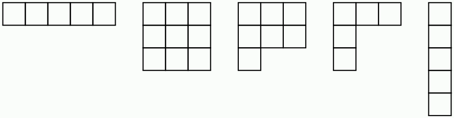

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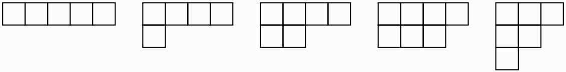

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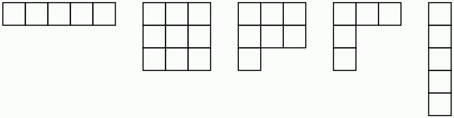

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Q Just coincidence? What about other partition theorems?

Euler's pentagonal number theorem

- Let $p_{d,e}(n)$ (respectively, $p_{d,o}(n)$) be the number of partitions of n into an even (respectively, odd) number of distinct parts.

EG
Euler

$$p_{d,e}(n) - p_{d,o}(n) = \begin{cases} (-1)^m, & \text{if } n = \frac{1}{2}m(3m \pm 1), \\ 0, & \text{otherwise.} \end{cases}$$

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- Likewise, let $q_{d,e}(n)$ (respectively, $q_{d,o}(n)$) be the number of partitions of perimeter n into an even (respectively, odd) number of distinct parts.

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Fu, Tang 2016

$$q_{d,e}(n) - q_{d,o}(n) = \begin{cases} (-1)^m, & \text{if } n = \frac{1}{2}(6m - 3 \pm 1), \\ 0, & \text{otherwise.} \end{cases}$$

Partitions of bounded perimeter

- The following very simple observation connects core partitions with partitions of bounded perimeter.

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COR An $(s, ds-1)$-core partition into distinct parts has perimeter at most $ds-2$.

Summary

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The number of (s, t)-core partitions is finite if and only if s and t are coprime. In that case, this number is

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Let $N_d(s)$ be the number of $(s, ds-1)$-core partitions into distinct parts. Then, $N_d(1) = 1$, $N_d(2) = d$ and

$$N_d(s) = N_d(s-1) + dN_d(s-2).$$

- In particular, there are F_s many $(s-1, s)$-core partitions into distinct parts,
- and 2^{s-1} many $(s, 2s-1)$-core partitions into distinct parts.

Q What is the number of (s, t)-core partitions into distinct parts in general?

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$s\backslash t$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	∞	2	∞	3	∞	4	∞	5	∞	6	∞
3	1	2	∞	3	4	∞	5	6	∞	7	8	∞
4	1	∞	3	∞	5	∞	8	∞	11	∞	15	∞
5	1	3	4	5	∞	8	16	18	16	∞	21	38
6	1	∞	∞	∞	8	∞	13	∞	∞	∞	32	∞
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12	1	∞	∞	∞	38	∞	114	∞	∞	∞	144	∞

Enumerating (s,t)-core partitions into distinct parts

Q What is the number of (s,t)-core partitions into distinct parts?

$s \backslash t$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	∞	2	∞	3	∞	4	∞	5	∞	6	∞
3	1	2	∞	3	4	∞	5	6	∞	7	8	∞
4	1	∞	3	∞	5	∞	8	∞	11	∞	15	∞
5	1	3	4	5	∞	8	16	18	16	∞	21	38
6	1	∞	∞	∞	8	∞	13	∞	∞	∞	32	∞
7	1	4	5	8	16	13	∞	21	64	50	64	114
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	…4	∞	101	∞
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	…	55	256	∞
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	…5	∞	89	∞
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	…56	89	∞	144
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	…	∞	144	∞

CONJ If s is odd, there are 2^{s-1} many $(s,s+2)$-core partitions into distinct parts.

Yan, Qin, Jin, Zhou (2016) have very recently proven this conjecture by analyzing order ideals in an associated poset introduced by Anderson.

Much simplified by Zaleski, Zeilberger (2016).

Enumerating (s,t)-core partitions into distinct parts

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3	1	2	∞	3	4	∞	5	6	∞	7	8	∞
4	1	∞	3	∞	5	∞	8	∞	11	∞	15	∞
5	1	3	4	5	∞	8	16	18	16	∞	21	38
6	1	∞	∞	∞	8	∞	13	∞	∞	∞	32	∞
7	1	4	5	8	16	13	∞	21	64	50	64	114
									4	∞	101	∞
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									5	∞	89	∞
									56	89	∞	144
										∞	144	∞

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$(s, s+3)$-core partitions into distinct parts

THM 2^{s-1} many $(s, s+2)$-core partitions into distinct parts (s odd).

Q How many $(s, s+3)$-core partitions into distinct parts?

- $1, 3, \infty, 8, 18, \infty, 50, 101, \infty, 291, 557, \infty, 1642, 3048, \infty, 9116, 16607, \ldots$

THM 2^{s-1} many $(s, s+2)$-core partitions into distinct parts (s odd).

- The largest size of $(2n-1, 2n+1)$-core partitions into distinct parts is
$$\frac{1}{24}n(n^2-1)(5n+6).$$
Now, also proven by Yan, Qin, Jin, Zhou (2016) and Zaleski, Zeilberger (2016).

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- $1, 3, \infty, 8, 18, \infty, 50, 101, \infty, 291, 557, \infty, 1642, 3048, \infty, 9116, 16607, \ldots$
- The largest size of $(3n-2, 3n+1)$-core partitions into distinct parts appears to be
$$\frac{1}{24}n(n^2-1)(9n+10).$$

$(s, s+3)$-core partitions into distinct parts

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- The largest size of $(3n-1, 3n+2)$-core partitions into distinct parts appears to be
$$\frac{1}{24} n(9n^3+38n^2+39n-14).$$

The size of a random core partition

DEF random variables

$X_{s,t}$: size of a (s,t)-core partition

$X_{s,t}^{(d)}$: size of a (s,t)-core partition into distinct parts

The size of a random core partition

DEF random variables

$X_{s,t}$: size of a (s,t)-core partition

$X_{s,t}^{(d)}$: size of a (s,t)-core partition into distinct parts

EG

$$E(X_{s,t}) = \frac{(s-1)(t-1)(s+t+1)}{24}$$

conjectured by Armstrong
first proved by Johnson

For comparison, largest size is $\frac{1}{24}(s^2-1)(t^2-1)$. (Olsson and Stanton, 2007)

EG

$$E(X_{s,s+1}^{(d)}) = \frac{1}{F_{s+1}} \sum_{i+j+k=s+1} F_i F_j F_k = \frac{1}{50F_{s+1}} \left((5s-6)sF_{s+1} - 6(s+1)F_s\right)$$

conjectured by Amdeberhan
first proved by Xiong

EG

$$E(X_{s,s+2}^{(d)}) = \frac{1}{128}\left((s-1)(5s^2+17s+16)\right)$$

Zaleski-Zeilberger

The size of a random core partition

DEF random variables

$X_{s,t}$: size of a (s,t)-core partition

$X_{s,t}^{(d)}$: size of a (s,t)-core partition into distinct parts

- Zeilberger (2015): explicit moments for $X_{s,t}$
- Zaleski (2016): explicit moments for $X_{s,s+1}^{(d)}$
- Zaleski-Zeilberger (2016): explicit moments for $X_{s,s+2}^{(d)}$

CONJ Zeilberger

Centralizing and standardizing, the distribution of $X_{s,t}$ as $s,t \to \infty$ with $s-t$ fixed agrees with the one of

$$\frac{1}{4\pi^2} \sum_{n=1}^{\infty} \frac{A_n^2 + B_n^2}{n^2}, \qquad A_n, B_n \text{ independent, } N(0,1).$$

CONJ Zaleski

The limiting distribution of $X_{s,s+1}^{(d)}$ is normal.

Q Zaleski Zeilberger

The limiting distribution of $X_{s,s+2}^{(d)}$ is not normal. What is it?

Enumerating (s,t)-core partitions into odd parts

Q What is the number of (s,t)-core partitions into odd parts?

$s\backslash t$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2	2	2
3	1	2	∞	4	4	∞	6	6	∞	8	8	∞
4	1	2	4	∞	7	6	9	∞	11	10	13	∞
5	1	2	4	7	∞	17	12	17	25	∞	41	31
6	1	2	∞	6	17	∞	31	21	∞	34	62	∞
7	1	2	6	9	12	31	∞	80	43	78	87	97
8	1	2	6	∞	17	21	80	∞	152	78	124	∞
9	1	2	∞	11	25	∞	43	152	∞	404	166	∞
10	1	2	8	10	∞	34	78	78	404	∞	790	308
11	1	2	8	13	41	62	87	124	166	790	∞	2140
12	1	2	∞	∞	31	∞	97	∞	∞	308	2140	∞

Enumerating (s,t)-core partitions into odd parts

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A modular supercongruence for ${}_6F_5$: An Apéry-like story

$$ {}_6F_5\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1, 1, 1 \end{matrix} \middle| 1\right)_{p-1} \equiv b(p) \pmod{p^3} $$

Joint work with:

Robert Osburn
(University College Dublin)

Wadim Zudilin
(University of Newcastle/
Radboud Universiteit)

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 A(n+1) = (2n+1)(17n^2+17n+5)A(n) - n^3 A(n-1).$$

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$$(n+1)^3 A(n+1) = (2n+1)(17n^2+17n+5)A(n) - n^3 A(n-1).$$

THM Apéry '78: $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.

proof The same recurrence is satisfied by the "near"-integers

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right).$$

Then, $\frac{B(n)}{A(n)} \to \zeta(3)$. But too fast for $\zeta(3)$ to be rational. □

Hypergeometric series

EG Trivially, the Apéry numbers have the representation

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$
$$= {}_4F_3\left(\left.\begin{matrix} -n, -n, n+1, n+1 \\ 1, 1, 1 \end{matrix}\right| 1\right).$$

- Here, ${}_4F_3$ is a hypergeometric series:

$${}_pF_q\left(\left.\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right| z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^n}{n!}.$$

Hypergeometric series

EG Trivially, the Apéry numbers have the representation

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = {}_4F_3\left(\left. \begin{matrix} -n, -n, n+1, n+1 \\ 1, 1, 1 \end{matrix} \right| 1 \right).$$

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$$ {}_pF_q\left(\left. \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \right| z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^n}{n!}.$$

- Similary, we have the **truncated hypergeometric series**

$$ {}_pF_q\left(\left. \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \right| z \right)_M = \sum_{k=0}^{M} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^n}{n!}.$$

A first connection to modular forms

- The Apéry numbers $A(n)$ satisfy $1, 5, 73, 1145, \ldots$

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n \geqslant 0} A(n) \underbrace{\left(\frac{\eta^{12}(\tau)\eta^{12}(6\tau)}{\eta^{12}(2\tau)\eta^{12}(3\tau)}\right)^n}_{\text{modular function}}.$$

$1 + 5q + 13q^2 + 23q^3 + O(q^4)$ $\quad$ $q - 12q^2 + 66q^3 + O(q^4)$ $\quad$ $q = e^{2\pi i\tau}$

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$1 + 5q + 13q^2 + 23q^3 + O(q^4)$ $\quad$ $q - 12q^2 + 66q^3 + O(q^4)$ $\quad$ $q = e^{2\pi i\tau}$

EG As a consequence, with $z = \sqrt{1 - 34x + x^2}$,

$$\sum_{n\geqslant 0} A(n)x^n = \frac{17 - x - z}{4\sqrt{2}(1 + x + z)^{3/2}}\, {}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1\end{matrix}\middle| -\frac{1024x}{(1 - x + z)^4}\right).$$

- Context:

$f(\tau)$	modular form of (integral) weight k
$x(\tau)$	modular function
$y(x)$	such that $y(x(\tau)) = f(\tau)$

Then $y(x)$ satisfies a linear differential equation of order $k + 1$.

A second connection to modular forms

THM Ahlgren–Ono '00

For primes $p > 2$, the Apéry numbers satisfy

$$A\left(\frac{p-1}{2}\right) \equiv a(p) \pmod{p^2}$$

where $a(n)$ are the Fourier coefficients of the Hecke eigenform

$$\eta(2\tau)^4 \eta(4\tau)^4 = \sum_{n=1}^{\infty} a(n) q^n$$

of weight 4 for the modular group $\Gamma_0(8)$.

- conjectured by Beukers '87, and proved modulo p
- similar congruences modulo p for other Apéry-like numbers

The “super” in these congruences

Fourier coefficients $a(p)$

Apéry sequence $A(n)$

The “super” in these congruences

Fourier coefficients $a(p)$

$\downarrow\uparrow$

point counts on modular curves modulo p

$\downarrow\uparrow$

character sums

$\downarrow\uparrow$

Gaussian hypergeometric series

$\downarrow\uparrow$

harmonic sums

$\downarrow\uparrow$

truncated hypergeometric series

$\downarrow\uparrow$

Apéry sequence $A(n)$

The "super" in these congruences

Fourier coefficients $a(p)$

$\downarrow\uparrow$

point counts on modular curves modulo p

$\downarrow\uparrow$

character sums

$\downarrow\uparrow$

Gaussian hypergeometric series

$\downarrow\uparrow$

harmonic sums

$\downarrow\uparrow$

truncated hypergeometric series

$\downarrow\uparrow$

Apéry sequence $A(n)$

equalities

"easy" mod p

Kilbourn's extension of the Ahlgren–Ono supercongruence

THM
Kilbourn
2006

$$ {}_4F_3\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1\end{matrix}\middle| 1\right)_{p-1} \equiv a(p) \pmod{p^3}, $$

for primes $p > 2$. Again, $a(n)$ are the Fourier coefficients of

$$ \eta(2\tau)^4\eta(4\tau)^4 = \sum_{n=1}^{\infty} a(n)q^n. $$

Kilbourn's extension of the Ahlgren–Ono supercongruence

THM Kilbourn 2006

$$ {}_4F_3\left(\begin{matrix}\frac12, \frac12, \frac12, \frac12 \\ 1, 1, 1\end{matrix}\middle| 1\right)_{p-1} \equiv a(p) \pmod{p^3}, $$

for primes $p > 2$. Again, $a(n)$ are the Fourier coefficients of

$$ \eta(2\tau)^4\eta(4\tau)^4 = \sum_{n=1}^{\infty} a(n)q^n. $$

- This result proved the first of 14 related supercongruences conjectured by Rodriguez-Villegas (2001) between
  - truncated hypergeometric series ${}_4F_3$ and
  - Fourier coefficients of modular forms of weight 4.
- Despite considerable progress, 11 of these remain open.

  McCarthy (2010), Fuselier-McCarthy (2016) prove one each; McCarthy (2010) proves "half" of all 14.
- The 14 supercongruence conjectures were complemented with $4+4$ conjectures for ${}_2F_1$ and ${}_3F_2$.

A supercongruence for ${}_6F_5$

THM
OSZ 2017

$$ {}_6F_5\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1, 1, 1 \end{matrix} \middle| 1 \right)_{p-1} \equiv b(p) \pmod{p^3}, $$

for primes $p > 2$. Here, $b(n)$ are the Fourier coefficients of

$$ \eta(\tau)^8\eta(4\tau)^4 + 8\eta(4\tau)^{12} = \eta(2\tau)^{12} + 32\eta(2\tau)^4\eta(8\tau)^8 = \sum_{n=1}^{\infty} b(n)q^n, $$

the unique newform in $S_6(\Gamma_0(8))$.

A supercongruence for ${}_6F_5$

THM
OSZ 2017

$$ {}_6F_5\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1, 1, 1\end{matrix} \middle| 1\right)_{p-1} \equiv b(p) \pmod{p^3}, $$

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the unique newform in $S_6(\Gamma_0(8))$.

- Conjectured by Mortenson based on numerical evidence, which further suggests it holds modulo p^5.

A supercongruence for ${}_6F_5$

THM OSZ 2017

$$ {}_6F_5\left(\begin{matrix} \frac12, \frac12, \frac12, \frac12, \frac12, \frac12 \\ 1, 1, 1, 1, 1 \end{matrix} \middle| 1 \right)_{p-1} \equiv b(p) \pmod{p^3}, $$

for primes $p > 2$. Here, $b(n)$ are the Fourier coefficients of

$$ \eta(\tau)^8\eta(4\tau)^4 + 8\eta(4\tau)^{12} = \eta(2\tau)^{12} + 32\eta(2\tau)^4\eta(8\tau)^8 = \sum_{n=1}^{\infty} b(n)q^n, $$

the unique newform in $S_6(\Gamma_0(8))$.

- Conjectured by Mortenson based on numerical evidence, which further suggests it holds modulo p^5.
- A result of Frechette, Ono and Papanikolas expresses the $b(p)$ in terms of Gaussian hypergeometric functions.
- Osburn and Schneider determined the resulting Gaussian hypergeometric functions modulo p^3 in terms of sums involving harmonic sums.

A brief impression of the available ingredients

THM In terms of Gaussian hypergeometric series,

$$b(p) = -p^5 {}_6F_5(1) + p^4 {}_4F_3(1) + p^3 {}_2F_1(1) + p^2.$$

- Conjectured by Koike; proven by Frechette, Ono and Papanikolas (2004).
- Here, ϕ_p is the quadratic character mod p, ϵ_p the trivial character, and

$$ {}_{n+1}F_n(x) = {}_{n+1}F_n\left(\begin{matrix}\phi_p,\, \phi_p,\, \ldots,\, \phi_p \\ \epsilon_p,\, \ldots,\, \epsilon_p\end{matrix} \middle| \, x\right)_p,$$

the finite field version of

$$ {}_{n+1}F_n\left(\begin{matrix}\frac{1}{2},\, \frac{1}{2},\, \ldots,\, \frac{1}{2} \\ 1,\, \ldots,\, 1\end{matrix} \middle| \, x\right).$$

A brief impression of the available ingredients

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  the finite field version of

$$ {}_{n+1}F_n\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2} \\ 1, \ldots, 1 \end{matrix} \,\middle|\, x\right). $$

- Since $p^n {}_{n+1}F_n(x) \in \mathbb{Z}$, it follows easily that

$$ b(p) \equiv -p^5 {}_6F_5(1) \equiv {}_6F_5\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1, 1, 1 \end{matrix} \,\middle|\, 1\right)_{p-1} \pmod{p}. $$

A brief impression of the available ingredients, cont'd

THM Osburn Schneider 2009

For primes $p > 2$ and $\ell \geqslant 2$,

$$-p^{2\ell-1} {}_{2\ell}F_{2\ell-1}(1) \equiv p^2 X_\ell(p) + pY_\ell(p) + Z_\ell(p) \pmod{p^3}.$$

- With $m = (p-1)/2$, the right-hand sides are

$$Z_\ell(p) = {}_{2\ell}F_{2\ell-1}\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1, 1, 1 \end{matrix} \middle| 1 \right)_m,$$

A brief impression of the available ingredients, cont'd

THM Osburn Schneider 2009

For primes $p > 2$ and $\ell \geqslant 2$,

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$$Z_\ell(p) = {}_{2\ell}F_{2\ell-1}\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1, 1, 1\end{matrix}\middle| 1\right)_m,$$

$$Y_\ell(p) = \sum_{k=0}^{m} (-1)^{\ell k} \binom{m+k}{k}^\ell \binom{m}{k}^\ell (1 - \ell k(2H_k - H_{m+k} - H_{m-k}),$$

$$X_\ell(p) = \sum_{k=0}^{m} (-1)^{\ell k} \binom{m+k}{k}^\ell \binom{m}{k}^\ell (1 + 4\ell k(H_{m+k} - H_k) + 2\ell^2 k^2 (H_{m+k} - H_k)^2 - \ell k^2 (H_{m+k}^{(2)} - H_k^{(2)})\big).$$

A harmonic identity

THM

$$\sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2 \big(1 - 2k(2H_k - H_{n+k} - H_{n-k})\big) = 1$$

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- As Nesterenko (1996), consider the partial fraction decomposition

$$R(t) = \frac{\prod_{j=1}^{n}(t-j)^2}{\prod_{j=0}^{n}(t+j)^2} = \sum_{k=0}^{n} \left(\frac{A_k}{(t+k)^2} + \frac{B_k}{t+k} \right).$$

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- The residue sum theorem applied to $tR(t)$ implies:

$$\sum_{k=0}^{n} (A_k - kB_k) = \sum_{\text{finite poles } x} \operatorname{Res}_x tR(t) = -\operatorname{Res}_\infty tR(t) = 1$$

A harmonic identity

THM

$$\sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2 (1 - 2k(2H_k - H_{n+k} - H_{n-k})) = 1$$

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- Only needed modulo p^2 and $n = (p-1)/2$ for Kilbourn's congruence.

A harmonic congruence

- Using identities similarly obtained from partial fractions, the ${}_6F_5$ congruence can be reduced to:

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$$\sum_{k=0}^{n}(-1)^k\binom{n+k}{k}^3\binom{n}{k}^3\big(1-3k(2H_k-H_{n+k}-H_{n-k})\big) \equiv \sum_{k=0}^{n}\binom{n+k}{k}^2\binom{n}{k}^2 \pmod{p^2}$$

for primes $p > 2$ and $n = (p-1)/2$.

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$$\equiv\sum_{k=0}^{n}\binom{n+k}{k}^2\binom{n}{k}^2 \pmod{p^2}$$

for primes $p>2$ and $n=(p-1)/2$.

- While identities can (now) be verified algorithmically, no algorithms are available for proving such congruences.

Paule–Schneider harmonic sums

DEF Paule, Schneider 2003

$$C_\ell(n) = \sum_{k=0}^{n} \binom{n}{k}^{\ell} \bigl(1 - \ell k (H_k - H_{n-k})\bigr)$$

- These are integer sequences: $C_1(n) = 1$, $C_2(n) = 0$, $C_3(n) = (-1)^n$,

$$C_4(n) = (-1)^n \binom{2n}{n}, \quad C_5(n) = (-1)^n \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}$$

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LEM OSZ '17; Chu, De Donno '05

$$C_6(n) = (-1)^n \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \binom{2k}{n}$$

- Open question: are there single-sum hypergeometric expressions for $C_\ell(n)$ when $\ell \geqslant 7$?

Another Apéry supercongruence

LEM OSZ '17 For all odd primes p,

$$A\left(\frac{p-1}{2}\right) \equiv C_6\left(\frac{p-1}{2}\right) \pmod{p^2}.$$

- Modular parametrizations by weight 2 modular forms of level 6 and 7.
- In other words,

$$\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2 \equiv (-1)^n\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}\binom{2k}{n} \pmod{p^2}.$$

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- Proving this congruence is easy once we replace the right-hand side with

$$C_6(n) = \sum_{k=0}^{n} (-1)^k \binom{3n+1}{n-k} \binom{n+k}{k}^3.$$

- Again, let us lament the lack of an algorithmic approach to such congruences.

An irrational equality

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$$A(n) = \frac{(-1)^n}{2} \sum_{k=0}^{n} \binom{n+k}{n} \binom{2n-k}{n} \binom{n}{k}^4 \times \left(2 + (n-2k)(5H_k - 5H_{n-k} - H_{n+k} + H_{2n-k})\right)$$

An irrational equality

LEM

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- This arises from a construction of linear forms in $\zeta(3)$ due to Ball. If

$$\widehat{R}(t) = \frac{n!^2\,(2t+n)\prod_{j=1}^{n}(t-j)\cdot\prod_{j=1}^{n}(t+n+j)}{\prod_{j=0}^{n}(t+j)^4} = \sum_{k=0}^{n}\left(\frac{\widehat{A}_k}{(t+k)^4} + \frac{\widehat{B}_k}{(t+k)^3} + \frac{\widehat{C}_k}{(t+k)^2} + \frac{\widehat{D}_k}{t+k}\right),$$

then $\sum_{t=1}^{\infty} \widehat{R}(t) = u_n \zeta(3) + v_n$.

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$$= \sum_{k=0}^{n} \left(\frac{\widehat{A}_k}{(t+k)^4} + \frac{\widehat{B}_k}{(t+k)^3} + \frac{\widehat{C}_k}{(t+k)^2} + \frac{\widehat{D}_k}{t+k} \right),$$

  then $\sum_{t=1}^{\infty} \widehat{R}(t) = u_n \zeta(3) + v_n$.

- Remarkably, the linear forms agree with the ones obtained from Nesterenko's construction:

$$A(n) = \frac{1}{2} u_n = \frac{1}{2} \sum_{k=0}^{n} \widehat{B}_k$$

Outlook

- Can we extend the congruence

$$ {}_6F_5\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1, 1, 1 \end{matrix} \middle| 1\right)_{p-1} \equiv b(p) \pmod{p^3}, $$

  and show that it holds modulo p^5?

  Special relevance of p^3: by Weil's bounds, $|b(p)| < 2p^{5/2}$

- Can the algorithmic approaches for $A = B$ be adjusted to $A \equiv B$?

- Why do these supercongruences hold?

  Very promising explanation suggested by Roberts, Rodriguez-Villegas, Watkins (2017) in terms of gaps between Hodge numbers of an associated motive.

THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`

Armin Straub
Core partitions into distinct parts and an analog of Euler's theorem
European Journal of Combinatorics, Vol. 57, 2016, p. 40-49

Robert Osburn, Armin Straub and Wadim Zudilin
A modular supercongruence for $_6F_5$: An Apéry-like story
Preprint, 2017. arXiv:1701.04098