Core partitions into distinct parts and an analog of Euler's theorem

AMS Special Session on Partition Theory and Related Topics
AMS Joint Mathematics Meetings, Atlanta

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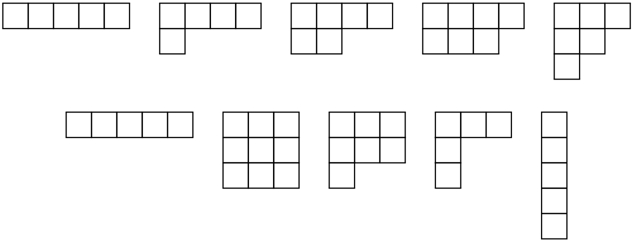
University of South Alabama



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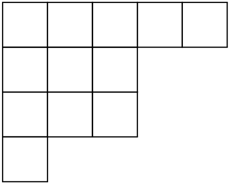

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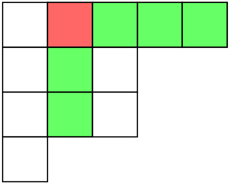

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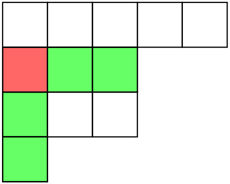

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LEM If a partition is t-core, then it is also rt-core for $r = 1, 2, 3 \ldots$

The number of core partitions

- Using the theory of modular forms, Granville and Ono (1996) showed:

  (The case $t = p$ of this completed the classification of simple groups with defect zero Brauer p-blocks.)

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- If $c_t(n)$ is the number of t-core partitions of n, then

$$\sum_{n=0}^{\infty} c_t(n) q^n = \prod_{n=1}^{\infty} \frac{(1-q^{tn})^t}{1-q^n}.$$

$$\sum_{n=0}^{\infty} c_2(n) q^n = \sum_{n=0}^{\infty} q^{\frac{1}{2}n(n+1)}, \quad \sum_{n=0}^{\infty} c_3(n) q^n = 1 + q + 2q^2 + 2q^4 + q^5 + 2q^6 + q^8 + \ldots$$

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COR The total number of t-core partitions is infinite.

Though this is probably the most complicated way possible to see that...

Counting core partitions

THM Anderson 2002: The number of (s,t)-core partitions is finite if and only if s and t are coprime.

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- Ford, Mai and Sze (2009) show that the number of self-conjugate (s,t)-core partitions is

$$\binom{\lfloor s/2\rfloor + \lfloor t/2\rfloor}{\lfloor s/2\rfloor}.$$

Core partitions into distinct parts

- Amdeberhan raises the interesting problem of counting the number of special partitions which are t-core for certain values of t.

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- He further conjectured that the largest possible size of an $(s, s+1)$-core partition into distinct parts is $\lfloor s(s+1)/6 \rfloor$, and that there is a unique such largest partition unless $s \equiv 1$ modulo 3, in which case there are two partitions of maximum size.
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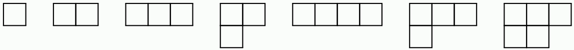
EG $s=5$, $F_6=8$ $\emptyset$



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- This special case was independently also proved by Xiong, who further shows the other claims by Amdeberhan.

A two-parameter generalization

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S 2016

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EG

The first few generalized Fibonacci polynomials $N_d(s)$ are

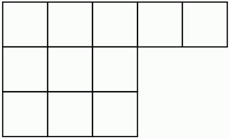
$$1, \quad d, \quad 2d, \quad d(d+2), \quad d(3d+2), \quad d(d^2+5d+2), \ldots$$

For $d = 1$, we recover the usual Fibonacci numbers.
For $d = 2$, we find $N_2(s) = 2^{s-1}$.

- Nice proof (and more!) via abaci structures by Nath and Sellers (2016).

The perimeter of a partition

DEF The **perimeter** of a partition is the maximum hook length in λ.

EG The partition  has perimeter 7.

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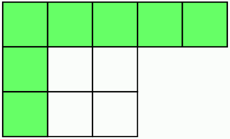
EG The partition  has perimeter 7.

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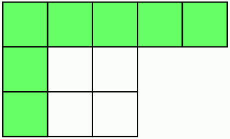
EG The partition  has perimeter 7.

- Introduced (up to a shift by 1) by Corteel and Lovejoy (2004) in their study of overpartitions.
- The perimeter is the largest part plus the number of parts (minus 1).
- The **rank** is the largest part minus the number of parts.

Euler's theorem and a simple analog

THM Euler

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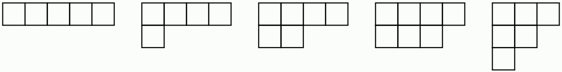
EG Partitions into distinct parts with perimeter 5:



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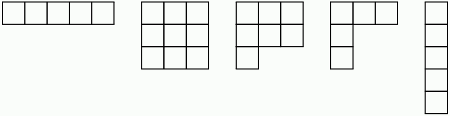

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THM
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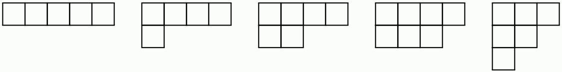

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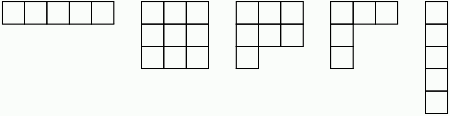

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- Many refinements of Euler's theorem are known.

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Q Just coincidence? What about other partition theorems?

Partitions of bounded perimeter

- The following very simple observation connects core partitions with partitions of bounded perimeter.

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COR An $(s, ds-1)$-core partition into distinct parts has perimeter at most $ds-2$.

Summary

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The number of (s,t)-core partitions is finite if and only if s and t are coprime. In that case, this number is

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Let $N_d(s)$ be the number of $(s, ds-1)$-core partitions into distinct parts. Then, $N_d(1)=1$, $N_d(2)=d$ and

$$N_d(s) = N_d(s-1) + dN_d(s-2).$$

- In particular, there are F_s many $(s-1,s)$-core partitions into distinct parts,
- and 2^{s-1} many $(s, 2s-1)$-core partitions into distinct parts.

Q What is the number of (s,t)-core partitions into distinct parts in general?

Enumerating (s,t)-core partitions into distinct parts

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$s\backslash t$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	∞	2	∞	3	∞	4	∞	5	∞	6	∞
3	1	2	∞	3	4	∞	5	6	∞	7	8	∞
4	1	∞	3	∞	5	∞	8	∞	11	∞	15	∞
5	1	3	4	5	∞	8	16	18	16	∞	21	38
6	1	∞	∞	∞	8	∞	13	∞	∞	∞	32	∞
7	1	4	5	8	16	13	∞	21	64	50	64	114
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CONJ If s is odd, there are 2^{s-1} many $(s,s+2)$-core partitions into distinct parts.

Yan, Qin, Jin, Zhou (2016) have very recently proven this conjecture by analyzing order ideals in an associated poset introduced by Anderson.

Much simplified by Zaleski, Zeilberger (2016).

Enumerating (s,t)-core partitions into distinct parts

Q What is the number of (s,t)-core partitions into distinct parts?

$s \backslash t$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	∞	2	∞	3	∞	4	∞	5	∞	6	∞
3	1	2	∞	3	4	∞	5	6	∞	7	8	∞
4	1	∞	3	∞	5	∞	8	∞	11	∞	15	∞
5	1	3	4	5	∞	8	16	18	16	∞	21	38
6	1	∞	∞	∞	8	∞	13	∞	∞	∞	32	∞
7	1	4	5	8	16	13	∞	21	64	50	64	114
									34	∞	101	∞
									∞	55	256	∞
									55	∞	89	∞
									56	89	∞	144
									∞	∞	144	∞

CONJ If s is odd, there are 2^{s-1} many $(s, s+2)$-core partitions into distinct parts.

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Much simplified by Zaleski, Zeilberger (2016).

$(s, s+3)$-core partitions into distinct parts

THM 2^{s-1} many $(s, s+2)$-core partitions into distinct parts (s odd).

Q How many $(s, s+3)$-core partitions into distinct parts?

- $1, 3, \infty, 8, 18, \infty, 50, 101, \infty, 291, 557, \infty, 1642, 3048, \infty, 9116, 16607, \ldots$

$(s, s+3)$-core partitions into distinct parts

THM 2^{s-1} many $(s, s+2)$-core partitions into distinct parts (s odd).

- The largest size of $(2n-1, 2n+1)$-core partitions into distinct parts is
$$\frac{1}{24} n(n^2-1)(5n+6).$$
Now, also proven by Yan, Qin, Jin, Zhou (2016) and Zaleski, Zeilberger (2016).

Q How many $(s, s+3)$-core partitions into distinct parts?

- $1, 3, \infty, 8, 18, \infty, 50, 101, \infty, 291, 557, \infty, 1642, 3048, \infty, 9116, 16607, \ldots$

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Q How many $(s, s+3)$-core partitions into distinct parts?

- $1, 3, \infty, 8, 18, \infty, 50, 101, \infty, 291, 557, \infty, 1642, 3048, \infty, 9116, 16607, \ldots$
- The largest size of $(3n-2, 3n+1)$-core partitions into distinct parts appears to be
$$\frac{1}{24}n(n^2-1)(9n+10).$$

THM 2^{s-1} many $(s, s+2)$-core partitions into distinct parts (s odd).

- The largest size of $(2n-1, 2n+1)$-core partitions into distinct parts is
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- The largest size of $(3n-2, 3n+1)$-core partitions into distinct parts appears to be
$$\frac{1}{24} n(n^2-1)(9n+10).$$
- The largest size of $(3n-1, 3n+2)$-core partitions into distinct parts appears to be
$$\frac{1}{24} n(9n^3+38n^2+39n-14).$$

The size of a random core partition

DEF random variables

$$X_{s,t} \quad : \quad \text{size of a } (s,t)\text{-core partition}$$

$$X_{s,t}^{(d)} \quad : \quad \text{size of a } (s,t)\text{-core partition into distinct parts}$$

The size of a random core partition

DEF random variables

$X_{s,t}$: size of a (s,t)-core partition

$X_{s,t}^{(d)}$: size of a (s,t)-core partition into distinct parts

EG

$$E(X_{s,t}) = \frac{(s-1)(t-1)(s+t+1)}{24}$$

conjectured by Armstrong
first proved by Johnson

For comparison, largest size is $\frac{1}{24}(s^2-1)(t^2-1)$. (Olsson and Stanton, 2007)

EG

$$E(X_{s,s+1}^{(d)}) = \frac{1}{F_{s+1}} \sum_{i+j+k=s+1} F_i F_j F_k = \frac{1}{50 F_{s+1}} \left((5s-6)sF_{s+1} - 6(s+1)F_s\right)$$

conjectured by Amdeberhan
first proved by Xiong

EG

$$E(X_{s,s+2}^{(d)}) = \frac{1}{128} \left((s-1)(5s^2+17s+16)\right)$$

Zaleski-Zeilberger

The size of a random core partition

DEF random variables

$X_{s,t}$: size of a (s,t)-core partition

$X_{s,t}^{(d)}$: size of a (s,t)-core partition into distinct parts

- Zeilberger (2015): explicit moments for $X_{s,t}$
- Zaleski (2016): explicit moments for $X_{s,s+1}^{(d)}$
- Zaleski-Zeilberger (2016): explicit moments for $X_{s,s+2}^{(d)}$

CONJ Zeilberger

Centralizing and standardizing, the distribution of $X_{s,t}$ as $s,t \to \infty$ with $s-t$ fixed agrees with the one of

$$\frac{1}{4\pi^2} \sum_{n=1}^{\infty} \frac{A_n^2 + B_n^2}{n^2}, \qquad A_n, B_n \text{ independent, } N(0,1).$$

CONJ Zaleski

The limiting distribution of $X_{s,s+1}^{(d)}$ is normal.

Q Zaleski Zeilberger

The limiting distribution of $X_{s,s+2}^{(d)}$ is not normal. What is it?

Enumerating (s,t)-core partitions into odd parts

Q What is the number of (s,t)-core partitions into odd parts?

$s\backslash t$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
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3	1	2	∞	4	4	∞	6	6	∞	8	8	∞
4	1	2	4	∞	7	6	9	∞	11	10	13	∞
5	1	2	4	7	∞	17	12	17	25	∞	41	31
6	1	2	∞	6	17	∞	31	21	∞	34	62	∞
7	1	2	6	9	12	31	∞	80	43	78	87	97
8	1	2	6	∞	17	21	80	∞	152	78	124	∞
9	1	2	∞	11	25	∞	43	152	∞	404	166	∞
10	1	2	8	10	∞	34	78	78	404	∞	790	308
11	1	2	8	13	41	62	87	124	166	790	∞	2140
12	1	2	∞	∞	31	∞	97	∞	∞	308	2140	∞

Enumerating (s, t)-core partitions into odd parts

Q What is the number of (s, t)-core partitions into odd parts?

$s \backslash t$	1	2	3	4	5	6	7	8	9	10	11	12
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3	1	2	∞	4	4	∞	6	6	∞	8	8	∞
4	1	2	4	∞	7	6	9	∞	11	10	13	∞
5	1	2	4	7	∞	17	12	17	25	∞	41	31
6	1	2	∞	6	17	∞	31	21	∞	34	62	∞
7	1	2	6	9	12	31	∞	80	43	78	87	97
8	1	2	6	∞	17	21	80	∞	152	78	124	∞
9	1	2	∞	11	25	∞	43	152	∞	404	166	∞
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12	1	2	∞	∞	31	∞	97	∞	∞	308	2140	∞

THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`

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