

# A modular supercongruence for ${}_6F_5$ : An Apéry-like story

Palmetto Number Theory Series (PANTS XXVIII)  
University of Tennessee

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**Armin Straub**

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University of South Alabama

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$${}_6F_5 \left( \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1, 1, 1 \end{matrix} \middle| 1 \right)_{p-1} \equiv b(p) \pmod{p^3}$$

**Joint work with:**



Robert Osburn  
(University College Dublin)



Wadim Zudilin  
(University of Newcastle/  
Radboud Universiteit)

- The **Apéry numbers**

1, 5, 73, 1445, ...

$$A(n) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 A(n+1) = (2n+1)(17n^2 + 17n + 5)A(n) - n^3 A(n-1).$$

# Apéry numbers and the irrationality of $\zeta(3)$

- The **Apéry numbers**

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$$(n+1)^3 A(n+1) = (2n+1)(17n^2 + 17n + 5)A(n) - n^3 A(n-1).$$

**THM**  
Apéry '78

$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$  is irrational.

**proof**

The same recurrence is satisfied by the “near”-integers

$$B(n) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 \left( \sum_{j=1}^n \frac{1}{j^3} + \sum_{m=1}^k \frac{(-1)^{m-1}}{2m^3 \binom{n}{m} \binom{n+m}{m}} \right).$$

Then,  $\frac{B(n)}{A(n)} \rightarrow \zeta(3)$ . But too fast for  $\zeta(3)$  to be rational.  $\square$

**EG** Trivially, the Apéry numbers have the representation

$$\begin{aligned} A(n) &= \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 \\ &= {}_4F_3 \left( \begin{matrix} -n, -n, n+1, n+1 \\ 1, 1, 1 \end{matrix} \middle| 1 \right). \end{aligned}$$

- Here,  ${}_4F_3$  is a hypergeometric series:

$${}_pF_q \left( \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}.$$

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- Similarly, we have the **truncated hypergeometric series**

$${}_pF_q \left( \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| z \right)_M = \sum_{k=0}^M \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}.$$

# A first connection to modular forms

- The Apéry numbers  $A(n)$  satisfy

1, 5, 73, 1145, ...

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n \geq 0} A(n) \underbrace{\left( \frac{\eta^{12}(\tau)\eta^{12}(6\tau)}{\eta^{12}(2\tau)\eta^{12}(3\tau)} \right)^n}_{\text{modular function}} \cdot$$

$1 + 5q + 13q^2 + 23q^3 + O(q^4)$        $q - 12q^2 + 66q^3 + O(q^4)$        $q = e^{2\pi i\tau}$

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$$1 + 5q + 13q^2 + 23q^3 + O(q^4) \qquad q - 12q^2 + 66q^3 + O(q^4) \qquad q = e^{2\pi i\tau}$$

**EG** As a consequence, with  $z = \sqrt{1 - 34x + x^2}$ ,

$$\sum_{n \geq 0} A(n)x^n = \frac{17 - x - z}{4\sqrt{2}(1 + x + z)^{3/2}} {}_3F_2 \left( \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} \middle| -\frac{1024x}{(1 - x + z)^4} \right).$$

**EG**  
S 2014 For contrast, the Apéry numbers are the diagonal coefficients of

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1x_2x_3x_4}.$$

# A second connection to modular forms

THM  
Ahlgren–  
Ono  
'00

For primes  $p > 2$ , the Apéry numbers satisfy

$$A\left(\frac{p-1}{2}\right) \equiv a(p) \pmod{p^2}$$

where  $a(n)$  are the Fourier coefficients of the Hecke eigenform

$$\eta(2\tau)^4 \eta(4\tau)^4 = \sum_{n=1}^{\infty} a(n) q^n$$

of weight 4 for the modular group  $\Gamma_0(8)$ .

- conjectured by Beukers '87, and proved modulo  $p$
- similar congruences modulo  $p$  for other Apéry-like numbers
- by the Deligne–Weil bounds,  $|a(p)| < 2p^{3/2}$ , the congruence determines the modular form

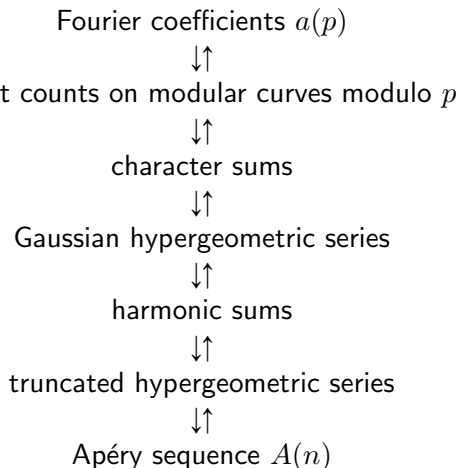


# The “super” in these congruences

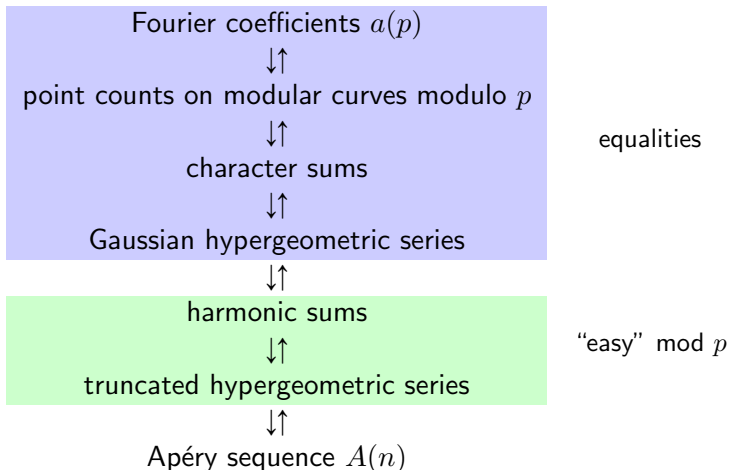
Fourier coefficients  $a(p)$

Apéry sequence  $A(n)$

# The “super” in these congruences



# The “super” in these congruences



THM  
Kilbourn  
2006

$${}_4F_3 \left( \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1 \end{matrix} \middle| 1 \right)_{p-1} \equiv a(p) \pmod{p^3},$$

for primes  $p > 2$ . Again,  $a(n)$  are the Fourier coefficients of

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- This result proved the first of 14 related supercongruences conjectured by Rodriguez-Villegas (2001) between
  - truncated hypergeometric series  ${}_4F_3$  and
  - Fourier coefficients of modular forms of weight 4.

- Despite considerable progress, 11 of these remain open.

McCarthy (2010), Fuselier–McCarthy (2016) prove one each; McCarthy (2010) proves “half” of each of the 14.

2017/5/4: Preprint by Long–Tu–Yui–Zudilin proving all 14 congruences.

- The 14 supercongruence conjectures were complemented with  $4 + 4$  conjectures for  ${}_2F_1$  and  ${}_3F_2$ .

# A supercongruence for ${}_6F_5$

THM  
OSZ  
2017

$${}_6F_5 \left( \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1, 1, 1 \end{matrix} \middle| 1 \right)_{p-1} \equiv b(p) \pmod{p^3},$$

for primes  $p > 2$ . Here,  $b(n)$  are the Fourier coefficients of

$$\eta(\tau)^8 \eta(4\tau)^4 + 8\eta(4\tau)^{12} = \sum_{n=1}^{\infty} b(n)q^n,$$

the unique newform in  $S_6(\Gamma_0(8))$ .

- Conjectured by Mortenson based on numerical evidence, which further suggests it holds modulo  $p^5$ .

# A supercongruence for ${}_6F_5$

THM  
OSZ  
2017

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the unique newform in  $S_6(\Gamma_0(8))$ .

- Conjectured by Mortenson based on numerical evidence, which further suggests it holds modulo  $p^5$ .
- A result of Frechette, Ono and Papanikolas expresses the  $b(p)$  in terms of Gaussian hypergeometric functions.
- Osburn and Schneider determined the resulting Gaussian hypergeometric functions modulo  $p^3$  in terms of sums involving harmonic sums.

# A brief impression of the available ingredients

**THM** In terms of Gaussian hypergeometric series,

$$b(p) = -p^5 {}_6F_5(1) + p^4 {}_4F_3(1) + p^3 {}_2F_1(1) + p^2.$$

- Conjectured by Koike; proven by Frechette, Ono and Papanikolas (2004).
- Here,  $\phi_p$  is the quadratic character mod  $p$ ,  $\epsilon_p$  the trivial character, and

$${}_{n+1}F_n(x) = {}_{n+1}F_n \left( \begin{matrix} \phi_p, \phi_p, \dots, \phi_p \\ \epsilon_p, \dots, \epsilon_p \end{matrix} \middle| x \right)_p,$$

the finite field version of

$${}_{n+1}F_n \left( \begin{matrix} \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2} \\ 1, \dots, 1 \end{matrix} \middle| x \right).$$



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- Since  $p^n {}_{n+1}F_n(x) \in \mathbb{Z}$ , it follows easily that

$$b(p) \equiv -p^5 {}_6F_5(1) \equiv {}_6F_5 \left( \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1, 1, 1 \end{matrix} \middle| 1 \right)_{p-1} \pmod{p}.$$

THM  
Osburn  
Schneider  
2009

For primes  $p > 2$  and  $\ell \geq 2$ ,

$$-p^{2\ell-1} {}_{2\ell}F_{2\ell-1}(1) \equiv Z_\ell(p) + pY_\ell(p) + p^2X_\ell(p) \pmod{p^3}.$$

- With  $m = (p-1)/2$ , the right-hand sides are

$$Z_\ell(p) = {}_{2\ell}F_{2\ell-1} \left( \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1, 1, 1 \end{matrix} \middle| 1 \right)_m,$$

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$$Y_\ell(p) = \sum_{k=0}^m (-1)^{\ell k} \binom{m+k}{k}^\ell \binom{m}{k}^\ell (1 - \ell k(2H_k - H_{m+k} - H_{m-k})),$$

$$X_\ell(p) = \sum_{k=0}^m (-1)^{\ell k} \binom{m+k}{k}^\ell \binom{m}{k}^\ell (1 + 4\ell k(H_{m+k} - H_k) \\ + 2\ell^2 k^2 (H_{m+k} - H_k)^2 - \ell k^2 (H_{m+k}^{(2)} - H_k^{(2)})).$$

# A harmonic identity and an irrational proof

THM

$$\sum_{k=0}^n \binom{n+k}{k}^2 \binom{n}{k}^2 (1 - 2k(2H_k - H_{n+k} - H_{n-k})) = 1$$

- This identity is needed to complete the proof the Ahlgren–Ono and Kilbourn supercongruences.



**Scott Ahlgren, Shalosh B. Ekhad, Ken Ono, Doron Zeilberger**

*A binomial coefficient identity associated to a conjecture of Beukers*

*Electronic Journal of Combinatorics*, Vol. 5, 1998, #R10

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- Nesterenko (1996) considered the rational function

$$R(t) = \frac{\prod_{j=1}^n (t-j)^2}{\prod_{j=0}^n (t+j)^2}$$

and showed that

$$\sum_{t=1}^{\infty} R'(t) = a_n \zeta(3) + b_n,$$

where  $a_n = -2A(n)$  and  $b_n \in \mathbb{Q}$ .

# A harmonic identity and an irrational proof

THM

$$\sum_{k=0}^n \binom{n+k}{k}^2 \binom{n}{k}^2 (1 - 2k(2H_k - H_{n+k} - H_{n-k})) = 1$$

- As Nesterenko (1996), consider the partial fraction decomposition

$$R(t) = \frac{\prod_{j=1}^n (t-j)^2}{\prod_{j=0}^n (t+j)^2} = \sum_{k=0}^n \left( \frac{A_k}{(t+k)^2} + \frac{B_k}{t+k} \right).$$

- One finds

$$A_k = \binom{n+k}{k}^2 \binom{n}{k}^2,$$
$$B_k = 2A_k (2H_k - H_{n+k} - H_{n-k}).$$

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- One finds

$$A_k = \binom{n+k}{k}^2 \binom{n}{k}^2,$$
$$B_k = 2A_k (2H_k - H_{n+k} - H_{n-k}).$$

- The residue sum theorem applied to  $tR(t)$  implies:

$$\sum_{k=0}^n (A_k - kB_k) = \sum_{\text{finite poles } x} \operatorname{Res}_x tR(t) = -\operatorname{Res}_\infty tR(t) = 1$$

# A harmonic congruence

- Using identities similarly obtained from partial fractions, the  ${}_6F_5$  congruence can be reduced to:

LEM  
OSZ  
2017

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n+k}{k}^3 \binom{n}{k}^3 (1 - 3k(2H_k - H_{n+k} - H_{n-k})) \\ & \equiv \sum_{k=0}^n \binom{n+k}{k}^2 \binom{n}{k}^2 \pmod{p^2} \end{aligned}$$

for primes  $p > 2$  and  $n = (p-1)/2$ .

- While identities can (now) be verified algorithmically, no algorithms are available for proving such congruences.



# An irrational equality

LEM

$$A(n) = \frac{(-1)^n}{2} \sum_{k=0}^n \binom{n+k}{n} \binom{2n-k}{n} \binom{n}{k}^4 \\ \times (2 + (n-2k)(5H_k - 5H_{n-k} - H_{n+k} + H_{2n-k}))$$

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- This arises from a construction of linear forms in  $\zeta(3)$  due to Ball. If

$$\hat{R}(t) = \frac{n!^2 (2t+n) \prod_{j=1}^n (t-j) \cdot \prod_{j=1}^n (t+n+j)}{\prod_{j=0}^n (t+j)^4} \\ = \sum_{k=0}^n \left( \frac{\hat{A}_k}{(t+k)^4} + \frac{\hat{B}_k}{(t+k)^3} + \frac{\hat{C}_k}{(t+k)^2} + \frac{\hat{D}_k}{t+k} \right),$$

$$\text{then } \sum_{t=1}^{\infty} \hat{R}(t) = u_n \zeta(3) + v_n.$$

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then  $\sum_{t=1}^{\infty} \hat{R}(t) = u_n \zeta(3) + v_n$ .

- Remarkably, these linear forms agree with Apéry's:

$$A(n) = \frac{1}{2} u_n = \frac{1}{2} \sum_{k=0}^n \hat{B}_k$$

- Can we extend the congruence

$${}_6F_5 \left( \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1, 1, 1 \end{matrix} \middle| 1 \right)_{p-1} \equiv b(p) \pmod{p^3},$$

and show that it holds modulo  $p^5$ ?

**Special relevance of  $p^3$ :** by the Deligne–Weil bounds,  $|b(p)| < 2p^{5/2}$

- Can the algorithmic approaches for  $A = B$  be adjusted to  $A \equiv B$ ?
- Why do these supercongruences hold?

Very promising explanation suggested by Roberts, Rodriguez-Villegas, Watkins (2017) in terms of gaps between Hodge numbers of an associated motive.

# THANK YOU!

Slides for this talk will be available from my website:  
<http://arminstraub.com/talks>



**Robert Osburn, Armin Straub and Wadim Zudilin**

*A modular supercongruence for  ${}_6F_5$ : An Apéry-like story*

Preprint, 2017. arXiv:1701.04098

DEF  
Paule,  
Schneider  
2003

$$C_\ell(n) = \sum_{k=0}^n \binom{n}{k}^\ell (1 - \ell k(H_k - H_{n-k}))$$

- These are integer sequences:  $C_1(n) = 1$ ,  $C_2(n) = 0$ ,  $C_3(n) = (-1)^n$ ,

$$C_4(n) = (-1)^n \binom{2n}{n}, \quad C_5(n) = (-1)^n \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}$$

**DEF**  
Paule,  
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**LEM**  
OSZ '17;  
Chu, De  
Donno  
'05

$$C_6(n) = (-1)^n \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k} \binom{2k}{n}$$

- Open question: are there single-sum hypergeometric expressions for  $C_\ell(n)$  when  $\ell \geq 7$ ?

# Another Apéry supercongruence

LEM  
OSZ '17

For all odd primes  $p$ ,

$$A \left( \frac{p-1}{2} \right) \equiv C_6 \left( \frac{p-1}{2} \right) \pmod{p^2}.$$

- Modular parametrizations by weight 2 modular forms of level 6 and 7.
- In other words,

$$\sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 \equiv (-1)^n \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k} \binom{2k}{n} \pmod{p^2}.$$



## Another Apéry supercongruence

LEM  
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- Proving this congruence is easy once we replace the right-hand side with

$$C_6(n) = \sum_{k=0}^n (-1)^k \binom{3n+1}{n-k} \binom{n+k}{k}^3.$$

- Again, let us lament the lack of an algorithmic approach to such congruences.