Special values of trigonometric Dirichlet series

Legacy of Ramanujan
OPSFA-13, NIST

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University of Illinois at Urbana–Champaign

$$\sum_{n=1}^{\infty} \frac{\sec^2(\pi n \sqrt{5})}{n^4} = \frac{14}{135} \pi^4$$

$$\sum_{n=1}^{\infty} \frac{\tan^3(\pi n \sqrt{6})}{n^5} = \frac{35,159}{17,820\sqrt{6}} \pi^4$$

Includes joint work with:

Bruce Berndt
University of Illinois at Urbana–Champaign

Rough outline

- examples of special values of trigonometric Dirichlet series
- main result on special values and outline of strategy
- just a brief comment on convergence
- introduction to Eichler integrals of Eisenstein series
- open problems (possibly unimodularity, if time permits)

Basic examples of trigonometric Dirichlet series

- Euler's identity:

$$\sum_{n=1}^{\infty} \frac{1}{n^{2m}} = -\frac{1}{2}(2\pi i)^{2m} \frac{B_{2m}}{(2m)!}$$

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- Half of the Clausen and Glaisher functions reduce, e.g.,

$$\sum_{n=1}^{\infty} \frac{\cos(\pi n \tau)}{n^{2m}} = \mathrm{poly}_m(\tau), \qquad \mathrm{poly}_1(\tau) = \frac{\pi^2}{12}\left(3\tau^2 - 6\tau + 2\right).$$

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- Ramanujan investigated trigonometric Dirichlet series of similar type. From his first letter to Hardy:

$$\sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^7} = \frac{19\pi^7}{56700}$$

In fact, this was already included in a general formula by Lerch.

One of Ramanujan's most famous formulas

THM Ramanujan, Grosswald

For $\alpha, \beta > 0$ such that $\alpha\beta = \pi^2$ and $m \in \mathbb{Z}$,

$$\alpha^{-m}\left\{\frac{\zeta(2m+1)}{2} + \sum_{n=1}^{\infty}\frac{n^{-2m-1}}{e^{2\alpha n}-1}\right\} = (-\beta)^{-m}\left\{\frac{\zeta(2m+1)}{2} + \sum_{n=1}^{\infty}\frac{n^{-2m-1}}{e^{2\beta n}-1}\right\}$$

$$-2^{2m}\sum_{n=0}^{m+1}(-1)^n\frac{B_{2n}}{(2n)!}\frac{B_{2m-2n+2}}{(2m-2n+2)!}\alpha^{m-n+1}\beta^n.$$

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- In terms of

$$\xi_s(\tau) = \sum_{n=1}^{\infty}\frac{\cot(\pi n\tau)}{n^s},$$

$$\frac{1}{e^x-1} = \tfrac{1}{2}\cot(\tfrac{x}{2}) - \tfrac{1}{2}$$

Ramanujan's formula takes the form

$$\tau^{2m-2}\xi_{2m-1}(-\tfrac{1}{\tau}) - \xi_{2m-1}(\tau) = (-1)^k(2\pi)^{2k-1}\sum_{s=0}^{k}\frac{B_{2s}}{(2s)!}\frac{B_{2k-2s}}{(2k-2s)!}\tau^{2s-1}.$$

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- Set $m = 4$ and $\tau = i$ to obtain $\displaystyle\sum_{n=1}^{\infty}\frac{\coth(\pi n)}{n^7} = \frac{19\pi^7}{56700}.$

EG
Ramanujan

$$\sum_{n=0}^{\infty} \frac{\tanh((2n+1)\pi/2)}{(2n+1)^3} = \frac{\pi^3}{32}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\operatorname{csch}(\pi n)}{n^3} = \frac{\pi^3}{360}$$

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EG Berndt 1976-78

$$\sum_{n=1}^{\infty} \frac{\cot\left(\pi n \frac{1+\sqrt{5}}{2}\right)}{n^3} = -\frac{\pi^3}{45\sqrt{5}}, \qquad \sum_{n=0}^{\infty} \frac{\tan(\pi(2n+1)\sqrt{5})}{(2n+1)^5} = \frac{23\pi^5}{3456\sqrt{5}}$$

THM Berndt 1976

Let $\tau = (a+b\sqrt{c})/2$ for $a, b, c \in \mathbb{Q}$ with $c > 0$ and $a^2 - cb^2 = 4\varepsilon$, $\varepsilon = \pm 1$. If $k > 1$,

$$\sum_{n=1}^{\infty} \frac{\cot(\pi n \tau)}{n^{2k-1}} = \frac{(-1)^{k-1}(2\pi)^{2k-1}}{1-\varepsilon\tau^{2k-2}} \sum_{s=0}^{k} \frac{B_{2s}}{(2s)!} \frac{B_{2k-2s}}{(2k-2s)!} \tau^{2s-1}.$$

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EG Komori-Matsumoto-Tsumura 2013

$$\sum_{n=1}^{\infty} \frac{\cot^2(\pi n \zeta_3)}{n^4} = -\frac{31}{2835}\pi^4, \qquad \sum_{n=1}^{\infty} \frac{\csc^2(\pi n \zeta_3)}{n^4} = \frac{1}{5670}\pi^4$$

(Here, ζ_3 is the primitive third root of unity.)

Secant zeta function

- Lalín, Rodrigue and Rogers introduce and study

$$\psi_s(\tau) = \sum_{n=1}^{\infty} \frac{\sec(\pi n \tau)}{n^s}.$$

- Clearly, $\psi_s(0) = \zeta(s)$. In particular, $\psi_2(0) = \frac{\pi^2}{6}$.

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LRR '13

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CONJ LRR '13 For positive integers m, r,

$$\psi_{2m}(\sqrt{r}) \in \mathbb{Q} \cdot \pi^{2m}.$$

- proof completed independently by Berndt–S and Charollois–Greenberg

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S 2014

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$$\sum_{n=1}^{\infty} \frac{\csc^2(\pi n\sqrt{11})}{n^4} = \frac{8}{385}\pi^4$$

$$\sum_{n=1}^{\infty} \frac{\sec^3(\pi n\sqrt{2})}{n^4} = -\frac{2483}{5220}\pi^4$$

$$\sum_{n=1}^{\infty} \frac{\tan^3(\pi n\sqrt{6})}{n^5} = \frac{35,159}{17,820\sqrt{6}}\pi^4$$

- For $a, b \in \mathbb{Z}$, let $\operatorname{trig}^{a,b} = \sec^a \csc^b$ be any product/quotient of trigonometric functions.

THM
S 2014

$$\sum_{n=1}^{\infty} \frac{\operatorname{trig}^{a,b}(\pi n \rho)}{n^s} \in \pi^s \mathbb{Q}(\rho)$$

provided that

- ρ is a real quadratic irrationality,
- $s \geqslant \max(a, b, 1) + 1$ (so that the series converges),
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- If, in addition, $\rho^2 \in \mathbb{Q}$ and $a + b \geqslant 0$, then the value is in $(\pi\rho)^s \mathbb{Q}$.

EG

$$\sum_{n=1}^{\infty} \frac{(\cos \cot)(\pi n \sqrt{2})}{n^3} = \left[\frac{1}{2} - \frac{253}{360\sqrt{2}}\right] \pi^3$$

(Here, $(a, b) = (-2, 1)$ does not satisfy $a + b \geqslant 0$.)

Strategy

- Rough strategy how to evaluate $\psi_s^{a,b}(\rho) = \sum_{n=1}^{\infty} \frac{\operatorname{trig}^{a,b}(\pi n \rho)}{n^s}$:
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  $(\operatorname{trig}^{a,b} = \sec^a \csc^b)$
  - Trivial case: $a \leqslant 0$ and $b \leqslant 0$. If $s > 1$ has the same parity as b, then
    $$\psi_s^{a,b}(\tau) = \pi^s f(\tau),$$
    where $f(\tau)$ is piecewise polynomial in τ with rational coefficients.

EG In terms of Bernoulli polynomials we have, for $0 < \tau < 1$,

$$\sum_{n=1}^{\infty} \frac{\cos(2\pi n\tau)}{n^{2m}} = \frac{(-1)^{m+1}}{2} \frac{(2\pi)^{2m}}{(2m)!} B_{2m}(\tau),$$
$$\sum_{n=1}^{\infty} \frac{\sin(2\pi n\tau)}{n^{2m+1}} = \frac{(-1)^{m+1}}{2} \frac{(2\pi)^{2m+1}}{(2m+1)!} B_{2m+1}(\tau).$$

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    $$\sec^2(\tau)\csc^2(\tau) = \sec^2(\tau) + \csc^2(\tau),$$

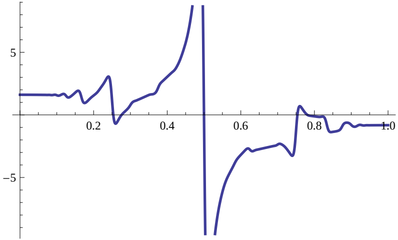
    and (here, a is odd)

    $$\sec^a(\tau) = \frac{1}{(a-1)!}(D^2 + (a-2)^2)(D^2 + (a-4)^2) \cdots (D^2 + 1^2)\sec(\tau),$$

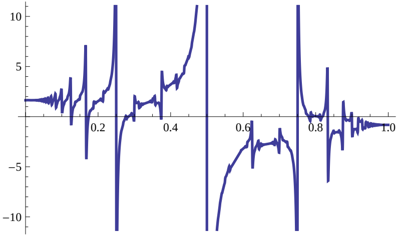
    to connect with the trivial and (derivatives of the) modular cases.

A glance at convergence

- $\psi_s(\tau) = \sum \dfrac{\sec(\pi n \tau)}{n^s}$ has singularity at rationals with even denominator



$\operatorname{Re}\ \psi_2(\tau + \varepsilon i)$ with $\varepsilon = 1/100$



$\operatorname{Re}\ \psi_2(\tau + \varepsilon i)$ with $\varepsilon = 1/1000$

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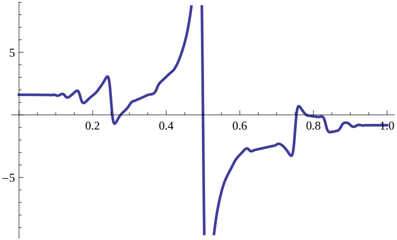

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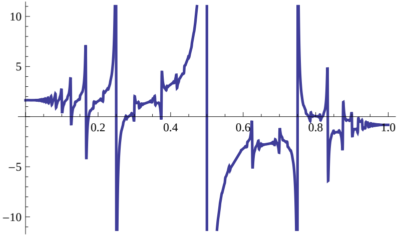

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THM Luca, Lalín–Rodrigue–Rogers 2013

The series $\psi_s(\tau) = \sum \frac{\sec(\pi n \tau)}{n^s}$ converges absolutely if

1. $\tau = p/q$ with q odd and $s > 1$,
2. τ is algebraic irrational and $s \geqslant 2$.

- Proof uses Thue–Siegel–Roth, as well as a result of Worley when $s = 2$ and τ is irrational

Ramanujan-type transformation formulas by residues

- Obviously, $\psi_s(\tau) = \sum \frac{\sec(\pi n \tau)}{n^s}$ satisfies $\psi_s(\tau+2) = \psi_s(\tau)$.

THM LRR, BS 2013

$$(1+\tau)^{2m-1}\psi_{2m}\left(\frac{\tau}{1+\tau}\right) - (1-\tau)^{2m-1}\psi_{2m}\left(\frac{\tau}{1-\tau}\right)$$
$$= \pi^{2m}\,\mathrm{rat}(\tau)$$

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proof Collect residues of the integral

$$I_C = \frac{1}{2\pi i}\int_C \frac{\sin(\pi\tau z)}{\sin(\pi(1+\tau)z)\sin(\pi(1-\tau)z)}\frac{\mathrm{d}z}{z^{s+1}}.$$

C are appropriate circles around the origin such that $I_C \to 0$ as $\mathrm{radius}(C) \to \infty$. $\square$

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$$\psi_2\left(\frac{\tau}{2\tau+1}\right) = \frac{1}{2\tau+1}\psi_2(\tau) + \pi^2\frac{\tau(3\tau^2+4\tau+2)}{6(2\tau+1)^2}$$

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- Hence, ψ_{2m} transforms under $T^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $R^2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$,
- Together, with $-I$, these two matrices generate $\Gamma(2)$.

Special values from transformation formulas

THM LRR, BS 2013 For any positive rational r,

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$$\psi_2\left(\frac{3\tau + 4}{2\tau + 3}\right) = -\frac{1}{2\tau + 3}\psi_2(\tau) - \frac{(\tau + 2)(3\tau^2 + 8\tau + 6)}{6(2\tau + 3)^2}\pi^2.$$

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- Hence, $\psi_2(\sqrt{2}) = -\dfrac{\pi^2}{3}$.

Modular forms

There's a saying attributed to Eichler that there are five fundamental operations of arithmetic: addition, subtraction, multiplication, division, and modular forms.

Andrew Wiles (BBC Interview, "The Proof", 1997)

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EG $\mathrm{SL}_2(\mathbb{Z})$

$$f(\tau + 1) = f(\tau), \qquad \tau^{-k} f(-1/\tau) = f(\tau).$$

EG $\mathrm{SL}_2(\mathbb{Z})$ **Eisenstein series** of weight $2k$:

$$G_{2k}(\tau) = \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m\tau+n)^{2k}}$$

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$$F|_{2-k}(\gamma-1) = \mathrm{poly}(\tau), \qquad \deg \mathrm{poly} \leqslant k-2.$$
- $\mathrm{poly}(\tau)$ is a **period polynomial** of the modular form f.
  The period polynomial encodes the critical L-values of f.

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- Differentiating the cotangent series $2k-1$ times, after using

$$\cot(\pi\tau) = \frac{1}{\pi}\sum_{j\in\mathbb{Z}} \frac{1}{\tau+j}, \qquad \lim_{N\to\infty}\sum_{j=-N}^{N}$$

we indeed get G_{2k}, up to a factor and the constant term.

Ramanujan's famous formula, again

THM Ramanujan, Grosswald

For $\alpha, \beta > 0$ such that $\alpha\beta = \pi^2$ and $m \in \mathbb{Z}$,

$$\alpha^{-m}\left\{\frac{\zeta(2m+1)}{2} + \sum_{n=1}^{\infty}\frac{n^{-2m-1}}{e^{2\alpha n}-1}\right\} = (-\beta)^{-m}\left\{\frac{\zeta(2m+1)}{2} + \sum_{n=1}^{\infty}\frac{n^{-2m-1}}{e^{2\beta n}-1}\right\}$$

$$-2^{2m}\sum_{n=0}^{m+1}(-1)^n\frac{B_{2n}}{(2n)!}\frac{B_{2m-2n+2}}{(2m-2n+2)!}\alpha^{m-n+1}\beta^n.$$

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- In terms of

$$\frac{1}{e^x-1} = \frac{1}{2}\cot\left(\frac{x}{2}\right) - \frac{1}{2}$$

$$\xi_s(\tau) = \sum_{n=1}^{\infty}\frac{\cot(\pi n\tau)}{n^s},$$

Ramanujan's formula takes the form

$$\tau^{2m-2}\xi_{2m-1}\left(-\frac{1}{\tau}\right) - \xi_{2m-1}(\tau) = (-1)^k(2\pi)^{2k-1}\sum_{s=0}^{k}\frac{B_{2s}}{(2s)!}\frac{B_{2k-2s}}{(2k-2s)!}\tau^{2s-1}.$$

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- Adjusting for the missing term in ξ_{2k-1}, the RHS is the period polynomial of the Eisenstein series G_{2k}.

Some open questions

- We have seen how to evaluate trigonometric series such as

$$\sum_{n=1}^{\infty} \frac{\sec^2(\pi n\sqrt{5})}{n^4} = \frac{14}{135}\pi^4.$$

- However, our method proceeds in a very recursive way. Can we give more explicit results or proofs?

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- However, our method proceeds in a very recursive way. Can we give more explicit results or proofs?
- In which cases can we evaluate more general series such as the following?

$$\sum_{n=1}^{\infty} \frac{\cot(\pi n\tau_1)\cdots\cot(\pi n\tau_r)}{n^s}$$

EG
Komori-Matsumoto-Tsumura 2013

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\csc(\pi n\zeta_5)\csc(\pi n\zeta_5^2)\cdots\csc(\pi n\zeta_5^4)}{n^6} = \frac{\pi^6}{935,550}$$

(Here, ζ_5 is the primitive fifth root of unity.)

THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`

B. Berndt, A. Straub
On a secant Dirichlet series and Eichler integrals of Eisenstein series
Preprint, 2014

A. Straub
Special values of trigonometric Dirichlet series and Eichler integrals
The Ramanujan Journal (special issue dedicated to Marvin Knopp), 2015

Unimodular polynomials

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EG Lehmer

$$x^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$$

has only the two real roots 0.850, 1.176 off the unit circle.

Lehmer's conjecture: $1.176\ldots$ is the smallest Mahler measure (greater than 1)

Ramanujan polynomials

- Following Gun–Murty–Rath, the **Ramanujan polynomials** are

$$R_k(X) = \sum_{s=0}^{k} \frac{B_s}{s!} \frac{B_{k-s}}{(k-s)!} X^{s-1}.$$

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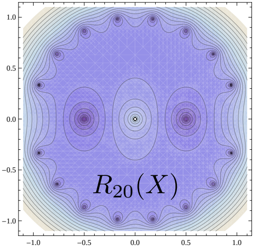
THM Murty-Smyth-Wang '11: All nonreal zeros of $R_k(X)$ lie on the unit circle. For $k \geqslant 2$, $R_{2k}(X)$ has exactly four real roots which approach $\pm 2^{\pm 1}$.



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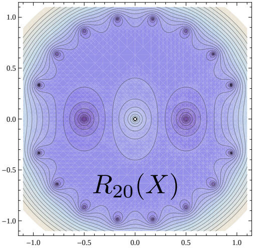
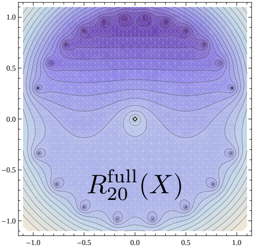
THM Lalín-Smyth '13: $R_{2k}(X) + \frac{\zeta(2k-1)}{(2\pi i)^{2k-1}}(X^{2k-2} - 1)$ is unimodular.





Unimodularity of period polynomials

THM Conrey-Farmer-Imamoglu 2012

For any Hecke cusp form (for $\mathrm{SL}_2(\mathbb{Z})$), the odd part of its period polynomial has

- trivial zeros at 0, ± 2, $\pm \frac{1}{2}$,
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THM El-Guindy–Raji 2013

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Q What about higher level?

- Consider the following **generalized Ramanujan polynomials**:

$$R_k(X;\chi,\psi)=\sum_{s=0}^{k}\frac{B_{s,\chi}}{s!}\frac{B_{k-s,\psi}}{(k-s)!}\left(\frac{X-1}{M}\right)^{k-s-1}(1-X^{s-1})$$

- Essentially, period polynomials: χ, ψ primitive, nonprincipal

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PROP
Berndt-S
2013

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PROP Berndt-S 2013

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CONJ If χ,ψ are nonprincipal real, then $R_k(X;\chi,\psi)$ is unimodular.

EG

$$R_k(X;\chi,1)$$

For χ real, conjecturally unimodular unless:

- $\chi = 1$: $R_{2k}(X;1,1)$ has real roots approaching $\pm 2^{\pm 1}$
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EG

$$R_k(X;1,\psi)$$

Conjecturally:

- unimodular for ψ one of
$$3-, 4-, 5+, 8\pm, 11-, 12+, 13+, 19-, 21+, 24+, \ldots$$
- all nonreal roots on the unit circle if ψ is one of
$$1+, 7-, 15-, 17+, 20-, 23-, 24-, \ldots$$
- four nonreal zeros off the unit circle if ψ is one of
$$35-, 59-, 83-, 131-, 155-, 179-, \ldots$$

- A second kind of **generalized Ramanujan polynomials**:

$$R_k(X) = \sum_{s=0}^{k} \frac{B_s}{s!} \frac{B_{k-s}}{(k-s)!} X^{s-1}$$

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CONJ If χ is nonprincipal real, then $S_k(X;\chi,\chi)$ is unimodular (up to trivial zero roots).