Supercongruences for Apéry-like numbers

AKLS seminar on Automorphic Forms
Universität zu Köln

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\[ A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \]

1, 5, 73, 1445, 33001, 819005, …

Includes joint work with:

Robert Osburn (University of Dublin)
Brundaban Sahu (NISER, India)
• Introducing Apéry-like numbers
  • they are integer solutions to certain three-term recurrences
  • are all of them known?
• Apéry-like numbers have interesting properties
  • connection to modular forms
  • supercongruences (still open in several cases)
  • multivariate extensions
  • polynomial analogs
• Apéry-like numbers occur in interesting places (if time permits)
  • moments of planar random walks
  • series for $1/\pi$
  • positivity of rational functions
  • counting points on algebraic varieties
  • ...
Apéry numbers and the irrationality of $\zeta(3)$

- The Apéry numbers $1, 5, 73, 1445, \ldots$

  \[
  A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2
  \]

  satisfy

  \[
  (n+1)^3 A(n+1) = (2n+1)(17n^2 + 17n + 5)A(n) - n^3 A(n-1).
  \]
Apéry numbers and the irrationality of $\zeta(3)$

- The Apéry numbers

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 A(n+1) = (2n+1)(17n^2 + 17n + 5)A(n) - n^3 A(n-1).$$

**Thm** Apéry '78

$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.

**Proof**

The same recurrence is satisfied by the “near”-integers

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left( \sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m} \binom{n+m}{m}} \right).$$

Then, $\frac{B(n)}{A(n)} \to \zeta(3)$. But too fast for $\zeta(3)$ to be rational.
Zagier’s search and Apéry-like numbers

- Recurrence for Apéry numbers is the case \((a, b, c) = (17, 5, 1)\) of

\[(n + 1)^3 u_{n+1} = (2n + 1)(an^2 + an + b)u_n - cn^3 u_{n-1}.

Are there other tuples \((a, b, c)\) for which the solution defined by 
\[u_{-1} = 0, \ u_0 = 1\] is integral?

---

Beukers, Zagier
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\]

Are there other tuples \((a, b, c)\) for which the solution defined by \(u_{-1} = 0, u_0 = 1\) is integral?

Essentially, only 14 tuples \((a, b, c)\) found. (Almkvist–Zudilin)

- 4 hypergeometric and 4 Legendrian solutions (with generating functions

\[
\binom{1}{2}, \alpha, 1 - \alpha \bigg| 4C_\alpha z \bigg), \quad \frac{1}{1 - C_\alpha z^2} F_1 \left( \alpha, 1 - \alpha \left| \frac{-C_\alpha z}{1 - C_\alpha z} \right. \right)^2,
\]

with \(\alpha = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\) and \(C_\alpha = 2^4, 3^3, 2^6, 2^4 \cdot 3^3\)

- 6 sporadic solutions

Similar (and intertwined) story for:

\[
(n + 1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2 u_{n-1}
\]

\[
(n + 1)^3 u_{n+1} = (2n + 1)(an^2 + an + b)u_n - n(cn^2 + d)u_{n-1}
\]

(Beukers, Zagier)

(Cooper)
The six sporadic Apéry-like numbers

<table>
<thead>
<tr>
<th>$(a, b, c)$</th>
<th>$A(n)$</th>
<th>Description</th>
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<tr>
<td>$(17, 5, 1)$</td>
<td>$\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$</td>
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</tr>
<tr>
<td>$(11, 5, 125)$</td>
<td>$\sum_k (-1)^k \binom{n}{k}^3 \left( \binom{4n-5k-1}{3n} + \binom{4n-5k}{3n} \right)$</td>
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<td></td>
</tr>
</tbody>
</table>
• The Apéry numbers $A(n)$ satisfy

\[
\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)} = \sum_{n \geq 0} A(n) \left( \frac{\eta^{12}(\tau)\eta^{12}(6\tau)}{\eta^{12}(2\tau)\eta^{12}(3\tau)} \right)^n.
\]

1 + 5q + 13q^2 + 23q^3 + O(q^4)

modular form

q − 12q^2 + 66q^3 + O(q^4)

modular function

q = e^{2\pi i \tau}
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FACT Not at all evidently, such a modular parametrization exists for all known Apéry-like numbers!
Apéry-like numbers and modular forms

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modular function

$$q = e^{2\pi i \tau}$$

FACT Not at all evidently, such a modular parametrization exists for all known Apéry-like numbers!

- As a consequence, with $z = \sqrt{1 - 34x + x^2}$,

$$\sum_{n \geq 0} A(n)x^n = \frac{17 - x - z}{4\sqrt{2}(1 + x + z)^{3/2}} \, _3F_2 \left( \begin{array}{ccc} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{array} \right| - \frac{1024x}{(1 - x + z)^4} \right).$$

- Context:

  $f(\tau)$ modular form of (integral) weight $k$

  $x(\tau)$ modular function

  $y(x)$ such that $y(x(\tau)) = f(\tau)$

Then $y(x)$ satisfies a linear differential equation of order $k + 1$. 

Supercongruences for Apéry-like numbers
• Chowla, Cowles, Cowles (1980) conjectured that, for primes $p \geq 5$,

$$A(p) \equiv 5 \pmod{p^3}.$$
Supercongruences for Apéry numbers

- Chowla, Cowles, Cowles (1980) conjectured that, for primes $p \geq 5$,
  \[ A(p) \equiv 5 \pmod{p^3}. \]
- Gessel (1982) proved that $A(mp) \equiv A(m) \pmod{p^3}$. 
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The Apéry numbers satisfy the **supercongruence** $(p \geq 5)$

\[ A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}. \]
Supercongruences for Apéry numbers

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The Apéry numbers satisfy the supercongruence \((p \geq 5)\)

\[ A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}. \]

For primes \( p \), simple combinatorics proves the congruence

\[ \binom{2p}{p} = \sum_k \binom{p}{k} \binom{p}{p-k} \equiv 1 + 1 \pmod{p^2}. \]

For \( p \geq 5 \), Wolstenholme’s congruence shows that, in fact,

\[ \binom{2p}{p} \equiv 2 \pmod{p^3}. \]
• Conjecturally, supercongruences like

\[ A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}} \]

hold for all Apéry-like numbers.

• Current state of affairs for the six sporadic sequences from earlier:

\[
\begin{array}{|c|c|c|}
\hline
(a, b, c) & A(n) & \text{Remarks} \\
\hline
(17, 5, 1) & \sum_k \binom{n}{k}^2 \binom{n+k}{n}^2 & \text{Beukers, Coster '87-'88} \\
(12, 4, 16) & \sum_k \binom{n}{k}^2 \binom{2k}{n}^2 & \text{Osburn–Sahu–S '14} \\
(10, 4, 64) & \sum_k \binom{n}{k}^2 \binom{2k}{n} \binom{2(n-k)}{n-k} & \text{Osburn–Sahu '11} \\
(7, 3, 81) & \sum_k (-1)^k 3^n - 3k \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3} & \text{open!! Amdeberhan '14} \\
(11, 5, 125) & \sum_k (-1)^k \binom{n}{k}^3 \left( \binom{4n-5k-1}{3n} + \binom{4n-5k}{3n} \right) & \text{Osburn–Sahu–S '14} \\
(9, 3, -27) & \sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k+l}{n} & \text{open} \\
\hline
\end{array}
\]
Non-super congruences are abundant

\[ a(mp^r) \equiv a(mp^{r-1}) \pmod{p^r} \quad (C) \]

- **realizable** sequences \(a(n)\), i.e., for some map \(T : X \to X\),

\[ a(n) = \# \{ x \in X : T^n x = x \} \]

\(\text{“points of period } n\)"

Everest–van der Poorten–Puri–Ward ’02, Arias de Reyna ’05
Non-super congruences are abundant

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- \( a(n) = ct \Lambda(x)^n \)

  if origin is only interior pt of the Newton polyhedron of \( \Lambda(x) \in \mathbb{Z}_p[x_1^{\pm 1}, \ldots, x_d^{\pm 1}] \)

  van Straten–Samol '09
Non-super congruences are abundant

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- \( a(n) = ct \Lambda(x)^n \) \hspace{1cm} \text{van Straten–Samol ’09}
  if origin is only interior pt of the Newton polyhedron of \( \Lambda(x) \in \mathbb{Z}_p[x_1^{\pm 1}, \ldots, x_d^{\pm 1}] \)

- If \( a(1) = 1 \), then (C) is equivalent to \( \exp \left( \sum_{n=1}^{\infty} \frac{a(n)}{n} T^n \right) \in \mathbb{Z}[[T]] \).
  This is a natural condition in **formal group theory**.
Cooper’s sporadic sequences

- Cooper’s search for integral solutions to

\[(n + 1)^3 u_{n+1} = (2n + 1)(an^2 + an + b)u_n - n(cn^2 + d)u_{n-1}\]

revealed three additional sporadic solutions:

\[s_{10}(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \binom{2k}{n}\]

\[s_{10}(n) = \sum_{k=0}^{n} \binom{n}{k}^4\]

\[s_{18}(n) = \sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k \binom{n}{k} \binom{2k}{k} \binom{2(n-k)}{n-k} \left[ \binom{2n-3k-1}{n} + \binom{2n-3k}{n} \right]\]
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\[s_7(mp) \equiv s_7(m) \pmod{p^3} \quad (p \geq 3)\]

\[s_{18}(mp) \equiv s_{18}(m) \pmod{p^2}\]
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- Cooper 2012

\[s_7(mp) \equiv s_7(m) \pmod{p^3} \quad p \geq 3\]

\[s_{18}(mp) \equiv s_{18}(m) \pmod{p^2}\]

- Osburn-Sahu-S 2014

\[s_7(mp^r) \equiv s_7(mp^{r-1}) \pmod{p^{3r}} \quad p \geq 5\]

\[s_{18}(mp^r) \equiv s_{18}(mp^{r-1}) \pmod{p^{2r}}\]
Diagonals

• Given a series

\[ F(x_1, \ldots, x_d) = \sum_{n_1, \ldots, n_d \geq 0} a(n_1, \ldots, n_d) x_1^{n_1} \cdots x_d^{n_d}, \]

its **diagonal coefficients** are the coefficients \( a(n, \ldots, n) \).

\[
\frac{1}{1 - x - y}
\]

has diagonal coefficients \( \binom{2n}{n} \).
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its \textbf{diagonal coefficients} are the coefficients \( a(n, \ldots, n) \).

\[ \frac{1}{1 - x - y} = \sum_{n=0}^{\infty} (x + y)^n \]

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For comparison, their univariate generating function is

\[
\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1 - 4x}}.
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• The diagonal of a rational function is \( D \)-finite.
The Apéry numbers are the diagonal coefficients of

\[
\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1x_2x_3x_4}
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\]

For \( x = (x_1, \ldots, x_n) \) and \( m = (m_1, \ldots, m_n) \in \mathbb{Z}_{\geq 0}^n \),

\[
[x^m] \frac{1}{\det(I_n - BX)} = [x^m] \prod_{i=1}^{n} \left( \sum_{j=1}^{n} B_{i,j} x_j \right)^{m_i},
\]

where \( B \in \mathbb{C}^{n \times n} \) and \( X \) is the diagonal matrix with entries \( x_1, \ldots, x_n \).
The Apéry numbers are the diagonal coefficients of

\[
\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1x_2x_3x_4} = \sum_{n \in \mathbb{Z}^4_{\geq 0}} A(n)x^n.
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\]

where \( B \in \mathbb{C}^{n \times n} \) and \( X \) is the diagonal matrix with entries \( x_1, \ldots, x_n \).

\[
B = \begin{pmatrix}
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 1 & 1 & 1
\end{pmatrix}, \quad X = \begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix}
\]

\[
A(n) = [x^n](x_1 + x_2 + x_3)^{n_1}(x_1 + x_2)^{n_2}(x_3 + x_4)^{n_3}(x_2 + x_3 + x_4)^{n_4}
\]
The Apéry numbers are the diagonal coefficients of

\[ \frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1x_2x_3x_4} = \sum_{n \in \mathbb{Z}^4 \geq 0} A(n) x^n. \]

- The coefficients are the multivariate Apéry numbers

\[ A(n) = \sum_{k \in \mathbb{Z}} \binom{n_1}{k} \binom{n_3}{k} \binom{n_1 + n_2 - k}{n_1} \binom{n_3 + n_4 - k}{n_3}. \]
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\]

- Univariate generating function:

\[
\sum_{n \geq 0} A(n) x^n = \frac{17 - x - z}{4\sqrt{2}(1 + x + z)^{3/2}} \; \binom{1}{2} \binom{1}{2} \binom{1}{2} 3F_2 \left( \begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{array} \right) - \frac{1024x}{(1 - x + z)^4},
\]

where \( z = \sqrt{1 - 34x + x^2} \).
The Apéry numbers are the diagonal coefficients of

\[
\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4} = \sum_{n \in \mathbb{Z}_4^2 \geq 0} A(n) x^n.
\]

- Well-developed theory of multivariate asymptotics
  e.g., Pemantle–Wilson

- Such diagonals are algebraic modulo $p^n$.
  Furstenberg, Deligne '67, '84

  Automatically leads to congruences such as

  \[
  A(n) \equiv \begin{cases} 
  1 \pmod{8}, & \text{if } n \text{ even}, \\
  5 \pmod{8}, & \text{if } n \text{ odd}.
  \end{cases}
  \]

  Chowla–Cowles–Cowles '80
  Rowland–Yassawi '13
Define $A(n) = A(n_1, n_2, n_3, n_4)$ by

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4} = \sum_{n \in \mathbb{Z}^4_{\geq 0}} A(n) x^n.$$

- The Apéry numbers are the diagonal coefficients.
- For $p \geq 5$, we have the multivariate supercongruences

$$A(np^r) \equiv A(np^{r-1}) \pmod{p^{3r}}.$$
Define $A(n) = A(n_1, n_2, n_3, n_4)$ by
\[
\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1x_2x_3x_4} = \sum_{n \in \mathbb{Z}^4_{\geq 0}} A(n)x^n.
\]

- The Apéry numbers are the diagonal coefficients.
- For $p \geq 5$, we have the multivariate supercongruences $A(np^r) \equiv A(np^{r-1}) \pmod{p^{3r}}$. 

\[
\sum_{n \geq 0} a(n)x^n = F(x) \implies \sum_{n \geq 0} a(np^n)x^{pn} = \frac{1}{p} \sum_{k=0}^{p-1} F(\zeta_p^k x)
\]

with $\zeta_p = e^{2\pi i / p}$.

- Hence, both $A(np^r)$ and $A(np^{r-1})$ have rational generating function. The proof, however, relies on an explicit binomial sum for the coefficients.
Define \( A(n) = A(n_1, n_2, n_3, n_4) \) by

\[
\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1x_2x_3x_4} = \sum_{n \in \mathbb{Z}^4_{\geq 0}} A(n) x^n.
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- The Apéry numbers are the diagonal coefficients.
- For \( p \geq 5 \), we have the \textit{multivariate supercongruences}

\[
A(np^r) \equiv A(np^{r-1}) \pmod{p^{3r}}.
\]

- By MacMahon’s Master Theorem,

\[
A(n) = \sum_{k \in \mathbb{Z}} \binom{n_1}{k} \binom{n_3}{k} \binom{n_1 + n_2 - k}{n_1} \binom{n_3 + n_4 - k}{n_3}.
\]
Define $A(n) = A(n_1, n_2, n_3, n_4)$ by
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- The Apéry numbers are the diagonal coefficients.
- For $p \geq 5$, we have the multivariate supercongruences
  \[
  A(np^r) \equiv A(np^{r-1} - 1) \pmod{p^{3r}}.
  \]

- By MacMahon’s Master Theorem,
  \[
  A(n) = \sum_{k \in \mathbb{Z}} \binom{n_1}{k} \binom{n_3}{k} \binom{n_1 + n_2 - k}{n_1} \binom{n_3 + n_4 - k}{n_3}.
  \]

- Because $A(n - 1) = A(-n, -n, -n, -n)$, we also find
  \[
  A(mp^r - 1) \equiv A(mp^{r-1} - 1) \pmod{p^{3r}}.
  \]
• Exhaustive search by Alin Bostan and Bruno Salvy:

\[ \frac{1}{1 - p(x, y, z, w)} \]

with \( p(x, y, z, w) \) a sum of distinct monomials; Apéry numbers as diagonal

\[
\begin{align*}
\frac{1}{1 - (x + y + xy)(z + w + zw)} & \quad \frac{1}{1 - (1 + w)(z + xy + yz + zx + xyz)} \\
\frac{1}{1 - (y + z + xy + xz + zw + xyw + xyzw)} & \quad \frac{1}{1 - (y + z + xz + wz + xyw + xzw + xyzw)} \\
\frac{1}{1 - (z + xy + yz + xw + xyw + yzw + xyzw)} & \quad \frac{1}{1 - (z + (x + y)(z + w) + xyz + xyzw)}
\end{align*}
\]
Many more conjectural multivariate supercongruences

- Exhaustive search by Alin Bostan and Bruno Salvy:

\[
\frac{1}{1 - p(x, y, z, w)} \text{ with } p(x, y, z, w) \text{ a sum of distinct monomials; Apéry numbers as diagonal}
\]

\[
\frac{1}{1 - (x + y + xy)(z + w + zw)}
\]

\[
\frac{1}{1 - (1 + w)(z + xy + yz + zx + xyz)}
\]

\[
\frac{1}{1 - (y + z + xy + xz + zw + xyw + xyzw)}
\]

\[
\frac{1}{1 - (y + z + xz + wz + xyw + xzw + xyzw)}
\]

\[
\frac{1}{1 - (z + xy + yz + xw + xyw + yzw + xyzw)}
\]

\[
\frac{1}{1 - (z + (x + y)(z + w) + xyz + xyzw)}
\]

The coefficients \( B(n) \) of each of these satisfy, for \( p \geq 5 \),

\[
B(n p^r) \equiv B(n p^{r-1}) \pmod{p^{3r}}.
\]
Let $\lambda \in \mathbb{Z}_0^\ell$ with $d = \lambda_1 + \ldots + \lambda_\ell$. Define $A_\lambda(n)$ by

$$\frac{1}{\prod_{1 \leq j \leq \ell} \left[ 1 - \sum_{1 \leq r \leq \lambda_j} x_{\lambda_1 + \ldots + \lambda_j - 1 + r} \right] - x_1 x_2 \cdots x_d} = \sum_{n \in \mathbb{Z}_0^d} A_\lambda(n) x^n.$$

- If $\ell \geq 2$, then, for all primes $p$,

$$A_\lambda(np^r) \equiv A_\lambda(np^{r-1}) \pmod{p^{2r}}.$$

- If $\ell \geq 2$ and $\max(\lambda_1, \ldots, \lambda_\ell) \leq 2$, then, for primes $p \geq 5$,

$$A_\lambda(np^r) \equiv A_\lambda(np^{r-1}) \pmod{p^{3r}}.$$

\[\begin{align*}
\lambda &= (2, 2) & \lambda &= (2, 1) \\
\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4} & \frac{1}{(1 - x_1 - x_2)(1 - x_3) - x_1 x_2 x_3} \\
\end{align*}\]
Further examples

EG

\[
\frac{1}{(1 - x_1 - x_2)(1 - x_3) - x_1 x_2 x_3}
\]

has as diagonal the Apéry-like numbers, associated with \( \zeta(2) \),

\[
B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}.
\]

EG

\[
\frac{1}{(1 - x_1)(1 - x_2) \cdots (1 - x_d) - x_1 x_2 \cdots x_d}
\]

has as diagonal the numbers

\[
Y_d(n) = \sum_{k=0}^{n} \binom{n}{k}^d.
\]

\[d = 3: \text{Franel}, d = 4: \text{Yang–Zudilin}\]

- In each case, we obtain supercongruences generalizing results of Coster (1988) and Chan–Cooper–Sica (2010).
The coefficients $Z(n)$ of

$$
\frac{1}{1 - (x_1 + x_2 + x_3 + x_4) + 27x_1x_2x_3x_4} = \sum_{n \in \mathbb{Z}_4 \geq 0} Z(n) x^n
$$

satisfy, for $p \geq 5$, the multivariate supercongruences

$$Z(np^r) \equiv Z(np^{r-1}) \pmod{p^{3r}}.
$$

- Here, the diagonal coefficients are the Almkvist–Zudilin numbers

$$Z(n) = \sum_{k=0}^{n} (-3)^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3},$$

for which the univariate congruences are still open.
The natural number $n$ has the $q$-analog:

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + \ldots + q^{n-1}$$

In the limit $q \to 1$ a $q$-analog reduces to the classical object.
The natural number $n$ has the $q$-analog:

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In the limit $q \to 1$ a $q$-analog reduces to the classical object.

The $q$-factorial:

$$[n]_q! = [n]_q [n - 1]_q \cdots [1]_q$$

The $q$-binomial coefficient:

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n - k]_q!} = \binom{n}{n - k}_q$$
A $q$-binomial coefficient

\[ \binom{6}{2} = \frac{6 \cdot 5}{2} = 3 \cdot 5 \]

\[ \binom{6}{2} = \frac{(1 + q + q^2 + q^3 + q^5)(1 + q + q^2 + q^3 + q^4)}{1 + q} \]
A $q$-binomial coefficient

\[
\binom{6}{2} = \frac{6 \cdot 5}{2} = 3 \cdot 5
\]

\[
\binom{6}{2}_q = \frac{(1 + q + q^2 + q^3 + q^5)(1 + q + q^2 + q^3 + q^4)}{1 + q}
\]

\[
= (1 - q + q^2) (1 + q + q^2) \binom{6}{2}_q = [3]_q [5]_q
\]
A $q$-binomial coefficient

\[
\binom{6}{2} = \frac{6 \cdot 5}{2} = 3 \cdot 5
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\]

\[
\binom{6}{2}_q = (1 - q + q^2)(1 + q + q^2)(1 + q + q^2 + q^3 + q^4)
\]

- $= \Phi_6(q)$
- $= [3]_q$
- $= [5]_q$

- The cyclotomic polynomial $\Phi_6(q)$ becomes 1 for $q = 1$
  and hence invisible in the classical world
The coefficients of $q$-binomial coefficients

- Here’s some $q$-binomials in expanded form:

\[
\begin{align*}
\binom{6}{2}_q &= q^8 + q^7 + 2q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1 \\
\binom{9}{3}_q &= q^{18} + q^{17} + 2q^{16} + 3q^{15} + 4q^{14} + 5q^{13} + 7q^{12} \\
&\quad\quad + 7q^{11} + 8q^{10} + 8q^9 + 8q^8 + 7q^7 + 7q^6 + 5q^5 \\
&\quad\quad + 4q^4 + 3q^3 + 2q^2 + q + 1
\end{align*}
\]

- The degree of the $q$-binomial is $k(n - k)$.
- All coefficients are positive!
- In fact, the coefficients are unimodal.

Sylvester, 1878
A few faces of the $q$-binomial coefficient

The $q$-binomial coefficient $\left(\begin{array}{c} n \\ k \end{array}\right)_q$

- satisfies a $q$-version of Pascal’s rule, $\left(\begin{array}{c} n \\ j \end{array}\right)_q = \left(\begin{array}{c} n-1 \\ j-1 \end{array}\right)_q + q^j \left(\begin{array}{c} n-1 \\ j \end{array}\right)_q$, 

- counts $k$-subsets of an $n$-set weighted by their sum,

- features in a binomial theorem for noncommuting variables, $\left(\begin{array}{c} x+y \\ n \end{array}\right) = \sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array}\right)_q x^j y^{n-j}$ if $x, y = qxy$,

- has a $q$-integral representation analogous to the beta function,

- counts the number of $k$-dimensional subspaces of $\mathbb{F}_n^q$. 
A few faces of the $q$-binomial coefficient

The $q$-binomial coefficient $\binom{n}{k}_q$

- satisfies a $q$-version of Pascal’s rule, $\binom{n}{j}_q = \binom{n-1}{j-1}_q + q^j \binom{n-1}{j}_q$,
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The \( q \)-binomial coefficient \( \binom{n}{k}_q \)

- satisfies a \( q \)-version of Pascal’s rule, \( \binom{n}{j}_q = \binom{n-1}{j-1}_q + q^j \binom{n-1}{j}_q \),
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- features in a binomial theorem for noncommuting variables,

\[
(x + y)^n = \sum_{j=0}^{n} \binom{n}{j}_q x^j y^{n-j}, \quad \text{if } yx = qxy,
\]
A few faces of the $q$-binomial coefficient

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\]

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- counts the number of $k$-dimensional subspaces of $\mathbb{F}_q^n$. 

A \( q \)-analog of Babbage’s congruence

- Combinatorially, we again obtain:

\[
\binom{2p}{p}_q = \sum_k \binom{p}{k}_q \binom{p}{p-k}_q (p-k)^2
\]

“\( q \)-Chu-Vandermonde”
A $q$-analog of Babbage's congruence

- Combinatorially, we again obtain:
  \[
  \binom{2p}{p}_q = \sum_k \binom{p}{k}_q \binom{p}{p-k}_q q^{(p-k)^2}
  \equiv q^{p^2} + 1 = [2]_{q^{p^2}}
  \]
  (Note that $[p]_q$ divides $\binom{p}{k}_q$ unless $k = 0$ or $k = p$.)

"$q$-Chu-Vandermonde"
A $q$-analog of Babbage’s congruence

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\]

\[
\equiv q^{p^2} + 1 = [2]_{q^{p^2}} \quad \text{(mod $[p]_q^2$)}
\]

(Note that $[p]_q$ divides $\binom{p}{k}_q$ unless $k = 0$ or $k = p$.)

- This combinatorial argument extends to show:

\[
\begin{array}{l}
\binom{ap}{bp}_q \equiv \binom{a}{b}_q q^{p^2} \\
\text{(mod $[p]_q^2$)}
\end{array}
\]
A $q$-analog of Babbage’s congruence

- Combinatorially, we again obtain: “$q$-Chu-Vandermonde”

\[
\binom{2p}{p}_q = \sum_k \binom{p}{k}_q \binom{p}{p-k}_q q^{(p-k)^2} \\
\equiv q^{p^2} + 1 = [2]_{q^p^2} \pmod{[p]_q^2}
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(Note that $[p]_q$ divides $\binom{p}{k}_q$ unless $k = 0$ or $k = p$.)

- This combinatorial argument extends to show:

\[
\binom{ap}{bp}_q \equiv \binom{a}{b}_q q^{p^2} \pmod{[p]_q^2}
\]

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- Similar results by Andrews; e.g.:

\[
\binom{ap}{bp}_q \equiv q^{(a-b)b(p^2)} \binom{a}{b}_q q^p \pmod{[p]_q^2}
\]
A $q$-analog of Ljunggren’s congruence

- The following answers the question of Andrews to find a $q$-analog of Wolstenholme’s congruence.

For any prime $p$,

$$
\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^{p^2}} - (a - b)b \binom{a}{b}_{q^{2p}} \frac{p^2 - 1}{24} (q^p - 1)^2 \pmod{[p]_q^3}.
$$
A \( q \)-analog of Ljunggren’s congruence

- The following answers the question of Andrews to find a \( q \)-analog of Wolstenholme’s congruence.

**THM S 2011**

For any prime \( p \),

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\]

**EG**

Choosing \( p = 13 \), \( a = 2 \), and \( b = 1 \), we have

\[
\binom{26}{13}_q = 1 + q^{169} - 14(q^{13} - 1)^2 + (1 + q + \ldots + q^{12})^3 f(q)
\]

where \( f(q) = 14 - 41q + 41q^2 - \ldots + q^{132} \) is an irreducible polynomial with integer coefficients.
A $q$-analog of Ljunggren’s congruence

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\[
\binom{ap}{bp}_q \equiv \binom{a}{b}^{q^2} - (a - b)b\binom{a}{b}\frac{p^2 - 1}{24}(q^p - 1)^2 \pmod{[p]_q^3}.
\]

- Note that $\frac{p^2 - 1}{24}$ is an integer if $(p, 6) = 1$.
  (The polynomial congruence holds for $p = 2, 3$ but coefficients are rational.)
A \( q \)-analog of Ljunggren’s congruence

- The following answers the question of Andrews to find a \( q \)-analog of Wolstenholme’s congruence.

For any prime \( p \),

\[
\left( \frac{a p}{b p} \right)_q \equiv \left( \frac{a}{b} \right) q^{p^2} - (a - b)b \left( \frac{a}{b} \right) \frac{p^2 - 1}{24} (q^p - 1)^2 \pmod{[p]^3}.
\]

- Note that \( \frac{p^2 - 1}{24} \) is an integer if \( (p, 6) = 1 \).

  (The polynomial congruence holds for \( p = 2, 3 \) but coefficients are rational.)

- Ljunggren’s classical congruence holds modulo \( p^{3+r} \) with \( r \) the \( p \)-adic valuation of \( ab(a - b) \left( \frac{a}{b} \right) \).

  Jacobsthal ’52

Is there a nice explanation or analog in the \( q \)-world?
The following answers the question of Andrews to find a $q$-analog of Wolstenholme’s congruence.

For any prime $p$,

\[
\left( \frac{ap}{bp} \right)_q \equiv \left( \frac{a}{b} \right)_{q^{p^2}} - (a - b)b \left( \frac{a}{b} \right) \frac{p^2 - 1}{24} (q^p - 1)^2 \pmod{[p]^3_q}.
\]

Note that $\frac{p^2 - 1}{24}$ is an integer if $(p, 6) = 1$.

(The polynomial congruence holds for $p = 2, 3$ but coefficients are rational.)

Ljunggren’s classical congruence holds modulo $p^{3+r}$ with $r$ the $p$-adic valuation of $ab(a - b) \left( \frac{a}{b} \right)$.

Is there a nice explanation or analog in the $q$-world?

The congruence holds mod $\Phi_n(q)^3$ if $p$ is replaced by any integer $n$.

(No classical counterpart since $\Phi_n(1) = 1$ unless $n$ is a prime power.)
A \textit{q}-version of the Apéry numbers

- A symmetric \textit{q}-analog of the Apéry numbers:

\[
A_q(n) = \sum_{k=0}^{n} q^{(n-k)^2} \binom{n}{k}^2 \binom{n+k}{k}^2
\]

- Appear implicitly in work of Krattenthaler–Rivoal–Zudilin ’06

The first few values are:

- \(A_q(0) = 1\)
- \(A_q(1) = 1 + 3q + q^2\)
- \(A_q(2) = 1 + 3q + 9q^2 + 14q^3 + 19q^4 + 14q^5 + 9q^6 + 3q^7 + q^8\)
- \(A_q(3) = 1 + 3q + 9q^2 + 22q^3 + 43q^4 + 76q^5 + 117q^6 + \ldots + 3q^{17} + q^{18}\)
A \( q \)-version of the Apéry numbers

- A symmetric \( q \)-analog of the Apéry numbers:

\[
A_q(n) = \sum_{k=0}^{n} q^{(n-k)^2} \left( n \right)_q^2 \left( \frac{n+k}{k} \right)_q^2
\]

- Appear implicitly in work of Krattenthaler–Rivoal–Zudilin ’06
- The first few values are:

\[
\begin{align*}
A(0) &= 1 & A_q(0) &= 1 \\
A(1) &= 5 & A_q(1) &= 1 + 3q + q^2 \\
A(2) &= 73 & A_q(2) &= 1 + 3q + 9q^2 + 14q^3 + 19q^4 + 14q^5 \\
& & & + 9q^6 + 3q^7 + q^8 \\
A(3) &= 1445 & A_q(3) &= 1 + 3q + 9q^2 + 22q^3 + 43q^4 + 76q^5 \\
& & & + 117q^6 + \ldots + 3q^{17} + q^{18}
\end{align*}
\]
The $q$-Apéry numbers, defined as

$$A_q(n) = \sum_{k=0}^{n} q^{(n-k)^2} \binom{n}{k}_q \binom{n+k}{k}_q^2,$$

describe the supercongruences

$$A_q(pn) \equiv A_{q^{p^2}}(n) - \frac{p^2 - 1}{12} (q^p - 1)^2 f(n) \pmod{[p]_q^3}.$$
The $q$-Apéry numbers, defined as

$$A_q(n) = \sum_{k=0}^{n} q^{(n-k)^2} \binom{n}{k}^2 q^2 \binom{n+k}{k}^2,$$

satisfy the supercongruences

$$A_q(pn) \equiv A_{q^{p^2}}(n) - \frac{p^2 - 1}{12} (q^p - 1)^2 f(n) \pmod{[p]_q^3}.$$

- The numbers $f(n)$ can be expressed as

$$f(n) = \sum_{k=0}^{n} g(n, k) \binom{n}{k}^2 \binom{n+k}{k}^2, \quad g(n, k) = k(2n-k) + \frac{k^4}{(n+k)^2}.$$

- Similar $q$-analogs and congruences for other Apéry-like numbers?
Some of many open problems

- Supercongruences for all Apéry-like numbers
  - proof of all the classical ones
  - uniform explanation, proofs not relying on binomial sums
- Apéry-like numbers as diagonals
  - find minimal rational functions
  - extend supercongruences
  - any structure?
- Polynomial analogs of Apéry-like numbers
  - find \( q \)-analogs (e.g., for Almkvist–Zudilin sequence)
  - \( q \)-supercongruences
  - is there a geometric picture?
- Many further questions remain.
  - is the known list complete?
  - higher-order analogs, Calabi–Yau DEs
  - modular supercongruences

\[ A \left( \frac{p - 1}{2} \right) \equiv a(p) \pmod{p^2}, \quad \sum_{n=1}^{\infty} a(n)q^n = \eta^4(2\tau)\eta^4(4\tau) \]

- \( \ldots \)
THANK YOU!

Slides for this talk will be available from my website:
http://arminstraub.com/talks

A. Straub
Multivariate Apéry numbers and supercongruences of rational functions

R. Osburn, B. Sahu, A. Straub
Supercongruences for sporadic sequences

A. Straub, W. Zudilin
Positivity of rational functions and their diagonals
to appear in Journal of Approximation Theory (special issue dedicated to Richard Askey), 2014

M. Rogers, A. Straub
A solution of Sun’s $520 \text{ challenge concerning} \ 520/\pi$
International Journal of Number Theory, Vol. 9, Nr. 5, 2013, p. 1273-1288

J. Borwein, A. Straub, J. Wan, W. Zudilin (appendix by D. Zagier)
Densities of short uniform random walks
Canadian Journal of Mathematics, Vol. 64, Nr. 5, 2012, p. 961-990

A. Straub
A $q$-analog of Ljunggren’s binomial congruence
DMTCS Proceedings: FPSAC 2011, p. 897-902
Applications of Apéry-like numbers

• Random walks

• Series for $1/\pi$

• Positivity of rational functions
Example 1: Random walks

$n$ steps in the plane
(length 1, random direction)
Example I: Random walks

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Example I: Random walks

\( n \) steps in the plane
(length 1, random direction)

- \( p_n(x) \) — probability density of distance traveled

\[ W_n(s) = \int_0^\infty x^s p_n(x) \, dx \]

- Classical Borwein–Nuyens–S–Wan, 2010

Supercongruences for Apéry-like numbers

Armin Straub

29 / 38
Example I: Random walks

\(n\) steps in the plane (length 1, random direction)

- \(p_n(x)\) — probability density of distance traveled

\[
W_n(s) = \int_0^\infty x^s p_n(x) \, dx
\]

- \(W_n(s)\) — probability moments

\[
W_2(1) = \frac{4}{\pi}, \quad W_3(1) = \frac{3}{16} \frac{2^{1/3}}{\pi^4} \Gamma^6\left(\frac{1}{3}\right) + \frac{27}{4} \frac{2^{2/3}}{\pi^4} \Gamma^6\left(\frac{2}{3}\right)
\]

classical

Borwein–Nuyens–S–Wan, 2010
Example 1: Random walks

- The probability moments

\[ W_n(s) = \int_0^\infty x^s p_n(x) \, dx \]

include the Apéry-like numbers

\[ W_3(2k) = \sum_{j=0}^k \binom{k}{j}^2 \binom{2j}{j}, \]

\[ W_4(2k) = \sum_{j=0}^k \binom{k}{j}^2 \binom{2j}{j} \binom{2(k - j)}{k - j}. \]
Example I: Random walks

- The probability moments

\[ W_n(s) = \int_0^\infty x^s p_n(x) \, dx \]

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\[ W_4(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} \binom{2(k-j)}{k-j}. \]
Example I: Random walks

\[ p_2(x) = \frac{2}{\pi \sqrt{4 - x^2}} \]

\[ p_3(x) = \frac{2\sqrt{3}}{\pi} \frac{x}{(3 + x^2)} \binom{2}{3} \binom{2}{3} \binom{x^2 (9 - x^2)^2}{(3 + x^2)^3} \]

\[ p_4(x) = \frac{2}{\pi^2} \frac{\sqrt{16 - x^2}}{x} \text{Re} \binom{3}{5} \binom{3}{5} \binom{(16 - x^2)^3}{108 x^4} \]

\[ p_5'(0) = \frac{\sqrt{5}}{40\pi^4} \Gamma\left(\frac{1}{15}\right) \Gamma\left(\frac{2}{15}\right) \Gamma\left(\frac{4}{15}\right) \Gamma\left(\frac{8}{15}\right) \approx 0.32993 \]
Example II: Series for $1/\pi$

\[
\frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{(1/2)^3_n}{n!^3} (6n + 1) \frac{1}{4^n}
\]

\[
\frac{16}{\pi} = \sum_{n=0}^{\infty} \frac{(1/2)^3_n}{n!^3} (42n + 5) \frac{1}{26^n}
\]

Srinivasa Ramanujan

*Modular equations and approximations to \(\pi\)*

Example II: Series for \(1/\pi\)

\[
\frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{(1/2)^3}{n!^3} (6n + 1) \frac{1}{4^n}
\]

\[
\frac{16}{\pi} = \sum_{n=0}^{\infty} \frac{(1/2)^3}{n!^3} (42n + 5) \frac{1}{26^n}
\]

\[
\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{n!^4} \frac{1103 + 26390n}{396^{4n}}
\]

- Last series used by Gosper in 1985 to compute 17,526,100 digits of \(\pi\)
- First proof of all of Ramanujan’s 17 series by Borwein brothers

\textbf{Srinivasa Ramanujan}

*Modular equations and approximations to \(\pi\)*


\textbf{Jonathan M. Borwein and Peter B. Borwein}

*Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity*

Wiley, 1987
Sato observed that series for $\frac{1}{\pi}$ can be built from Apéry-like numbers:

For the Domb numbers $D(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$,

$$\frac{8}{\sqrt{3\pi}} = \sum_{n=0}^{\infty} D(n) \frac{5n + 1}{2^{6n}}.$$
Example II: Series for \(1/\pi\)

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  For the Domb numbers \(D(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}\),

  \[
  \frac{8}{\sqrt{3\pi}} = \sum_{n=0}^{\infty} D(n) \frac{5n+1}{2^{6n}}.
  \]

- Sun offered a $520 bounty for a proof the following series:

  \[
  \frac{520}{\pi} = \sum_{n=0}^{\infty} \frac{1054n + 233}{480^n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} (-1)^k 8^{2k-n}.
  \]
Example II: Series for $\frac{1}{\pi}$

- Suppose we have a sequence $a_n$ with **modular parametrization**

$$\sum_{n=0}^{\infty} a_n x(\tau)^n = f(\tau).$$

- Modular function
- Modular form

- Then:

$$\sum_{n=0}^{\infty} a_n (A + Bn) x(\tau)^n = Af(\tau) + B \frac{x(\tau)}{x'(\tau)} f'(\tau)$$

$$\sum_{n=0}^{\infty} \frac{(1/2)^3}{n!^3} (42n + 5) \frac{1}{26n} = \frac{16}{\pi}$$
Example II: Series for $1/\pi$

- Suppose we have a sequence $a_n$ with modular parametrization

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- Then:

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$$\sum_{n=0}^{\infty} \frac{(1/2)^3 n!^3}{(42n + 5) 26n} = \frac{16}{\pi}$$

**FACT**

- For $\tau \in \mathbb{Q}(\sqrt{-d})$, $x(\tau)$ is an algebraic number.
- $f'(\tau)$ is a quasimodular form.
- Prototypical $E_2(\tau)$ satisfies $\tau^{-2} E_2(-\frac{1}{\tau}) - E_2(\tau) = \frac{6}{\pi i \tau}$.

- These are the main ingredients for series for $1/\pi$. Mix and stir.
Example III: Positivity of rational functions

- A rational function

\[ F(x_1, \ldots, x_d) = \sum_{n_1, \ldots, n_d \geq 0} a_{n_1, \ldots, n_d} x_1^{n_1} \cdots x_d^{n_d} \]

is positive if \( a_{n_1, \ldots, n_d} > 0 \) for all indices.

The following rational functions are positive.

\[ S(x, y, z) = \frac{1}{1 - (x + y + z) + \frac{3}{4}(xy + yz + zx)} \]

\[ A(x, y, z) = \frac{1}{1 - (x + y + z) + 4xyz} \]

- Both functions are on the boundary of positivity.
Example III: Positivity of rational functions

- A rational function

\[ F(x_1, \ldots, x_d) = \sum_{n_1, \ldots, n_d \geq 0} a_{n_1, \ldots, n_d} x_1^{n_1} \cdots x_d^{n_d} \]

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The following rational functions are positive.

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S(x, y, z) = \frac{1}{1 - (x + y + z) + \frac{3}{4}(xy + yz + zx)} \\
A(x, y, z) = \frac{1}{1 - (x + y + z) + 4xyz}
\]

- Both functions are on the boundary of positivity.
- The diagonal coefficients of \( A \) are the **Franel numbers**

\[
 \sum_{k=0}^{n} \binom{n}{k}^3
\]

Szegö '33
Kaluza '33
Askey–Gasper '72
S '08
Askey–Gasper '77
Koornwinder '78
Ismail–Tamhankar '79
Gillis–Reznick–Zeilberger '83
Example III: Positivity of rational functions

The following rational function is positive:

$$\frac{1}{1 - (x + y + z + w) + 2(yzw + xzw + xyw + xyz) + 4xyzw}.$$

- Would imply conjectured positivity of Lewy–Askey rational function

$$\frac{1}{1 - (x + y + z + w) + \frac{2}{3}(xy + xz + xw + yz + yw + zw)}.$$

Recent proof of non-negativity by Scott and Sokal, 2013
Example III: Positivity of rational functions

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Recent proof of non-negativity by Scott and Sokal, 2013

The Kauers–Zeilberger function has diagonal coefficients

$$d_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n}^2.$$
Consider rational functions $F = 1/p(x_1, \ldots, x_d)$ with $p$ a symmetric polynomial, linear in each variable.

**Q** Under what condition(s) is the positivity of $F$ implied by the positivity of its diagonal?

**EG**

- $\frac{1}{1 - (x + y)}$ is positive.
- $\frac{1}{1 + x + y}$ has positive diagonal but is not positive.
Consider rational functions $F = 1/p(x_1, \ldots, x_d)$ with $p$ a symmetric polynomial, linear in each variable.

Under what condition(s) is the positivity of $F$ implied by the positivity of its diagonal?

- $\frac{1}{1 - (x + y)}$ is positive.
- $\frac{1}{1 + x + y}$ has positive diagonal but is not positive.
- $\frac{1}{1 + x}$ is not positive.
• Consider rational functions \( F = 1/p(x_1, \ldots, x_d) \) with \( p \) a symmetric polynomial, linear in each variable.

Q Under what condition(s) is the positivity of \( F \) implied by the positivity of its diagonal?

EG

- \( \frac{1}{1 - (x + y)} \) is positive.
- \( \frac{1}{1 + x + y} \) has positive diagonal but is not positive.
- \( \frac{1}{1 + x} \) is not positive.

Q \( F \) positive \( \iff \) diagonal of \( F \) and \( F|_{x_d=0} \) are positive?
Consider rational functions $F = 1/p(x_1, \ldots, x_d)$ with $p$ a symmetric polynomial, linear in each variable.

Under what condition(s) is the positivity of $F$ implied by the positivity of its diagonal?

- $\frac{1}{1 - (x + y)}$ is positive.
- $\frac{1}{1 + x + y}$ has positive diagonal but is not positive.
- $\frac{1}{1 + x}$ is not positive.

$F$ positive $\iff$ diagonal of $F$ and $F|_{x_d=0}$ are positive?

$F(x, y) = \frac{1}{1 + c_1(x + y) + c_2 xy}$ is positive

$\iff$ diagonal of $F$ and $F|_{y=0}$ are positive
THANK YOU!

Slides for this talk will be available from my website:
http://arminstraub.com/talks

A. Straub
Multivariate Apéry numbers and supercongruences of rational functions

R. Osburn, B. Sahu, A. Straub
Supercongruences for sporadic sequences

A. Straub, W. Zudilin
Positivity of rational functions and their diagonals
to appear in Journal of Approximation Theory (special issue dedicated to Richard Askey), 2014

M. Rogers, A. Straub
A solution of Sun’s $520$ challenge concerning $520/\pi$
International Journal of Number Theory, Vol. 9, Nr. 5, 2013, p. 1273-1288

J. Borwein, A. Straub, J. Wan, W. Zudilin (appendix by D. Zagier)
Densities of short uniform random walks
Canadian Journal of Mathematics, Vol. 64, Nr. 5, 2012, p. 961-990