On a secant Dirichlet series and Eichler integrals of Eisenstein series

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Based on joint work with:

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- Lalín, Rodrigue and Rogers introduce and study

$$\psi_s(\tau) = \sum_{n=1}^{\infty} \frac{\sec(\pi n \tau)}{n^s}.$$

- Clearly, $\psi_s(0) = \zeta(s)$. In particular, $\psi_2(0) = \frac{\pi^2}{6}$.

EG LRR '13

$$\psi_2(\sqrt{2}) = -\frac{\pi^2}{3}, \qquad \psi_2(\sqrt{6}) = \frac{2\pi^2}{3}$$

CONJ LRR '13 For positive integers m, r,

$$\psi_{2m}(\sqrt{r}) \in \mathbb{Q} \cdot \pi^{2m}.$$

Basic examples of trigonometric Dirichlet series

- Euler's identity:

$$\sum_{n=1}^{\infty} \frac{1}{n^{2m}} = -\frac{1}{2}(2\pi i)^{2m} \frac{B_{2m}}{(2m)!}$$

- Half of the Clausen and Glaisher functions reduce, e.g.,

$$\sum_{n=1}^{\infty} \frac{\cos(\pi n \tau)}{n^{2m}} = \operatorname{poly}_m(\tau), \qquad \operatorname{poly}_1(\tau) = \frac{\pi^2}{12}\left(3\tau^2 - 6\tau + 2\right).$$

- Ramanujan investigated trigonometric Dirichlet series of similar type. From his first letter to Hardy:

$$\sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^7} = \frac{19\pi^7}{56700}$$

In fact, this was already included in a general formula by Lerch.

One of Ramanujan's most well-known formulas

THM Ramanujan, Grosswald

For $\alpha, \beta > 0$ such that $\alpha\beta = \pi^2$ and $m \in \mathbb{Z}$,

$$\alpha^{-m} \left\{ \frac{\zeta(2m+1)}{2} + \sum_{n=1}^{\infty} \frac{n^{-2m-1}}{e^{2\alpha n} - 1} \right\} = (-\beta)^{-m} \left\{ \frac{\zeta(2m+1)}{2} + \sum_{n=1}^{\infty} \frac{n^{-2m-1}}{e^{2\beta n} - 1} \right\}$$

$$-2^{2m} \sum_{n=0}^{m+1} (-1)^n \frac{B_{2n}}{(2n)!} \frac{B_{2m-2n+2}}{(2m-2n+2)!} \alpha^{m-n+1} \beta^n.$$

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- In terms of $\xi_s(\tau) = \sum \frac{\cot(\pi n \tau)}{n^s}$, Ramanujan's formula becomes

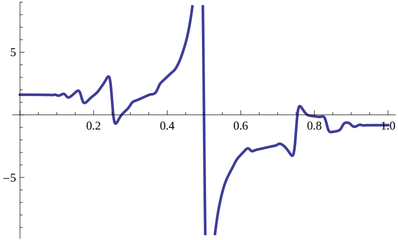
$$\xi_{2k-1}|_{2-2k}(S-1) = (-1)^k (2\pi)^{2k-1} \sum_{s=0}^{k} \frac{B_{2s}}{(2s)!} \frac{B_{2k-2s}}{(2k-2s)!} \tau^{2s-1}.$$

DEF slash operator

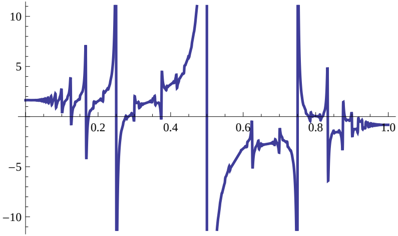
$$F|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau) = (c\tau + d)^{-k} F\left(\frac{a\tau + b}{c\tau + d} \right)$$

Secant zeta function: Convergence

- $\psi_s(\tau) = \sum \frac{\sec(\pi n \tau)}{n^s}$ has singularity at rationals with even denominator



Re $\psi_2(\tau + \varepsilon i)$ with $\varepsilon = 1/100$



Re $\psi_2(\tau + \varepsilon i)$ with $\varepsilon = 1/1000$

THM Lalín–Rodrigue–Rogers 2013

The series $\psi_s(\tau) = \sum \frac{\sec(\pi n \tau)}{n^s}$ converges absolutely if

1. $\tau = p/q$ with q odd and $s > 1$,
2. τ is algebraic irrational and $s \geqslant 2$.

- Proof uses Thue–Siegel–Roth, as well as a result of Worley when $s = 2$ and τ is irrational

Secant zeta function: Functional equation

- Obviously, $\psi_s(\tau) = \sum \frac{\sec(\pi n \tau)}{n^s}$ satisfies $\psi_s(\tau + 2) = \psi_s(\tau)$.

THM LRR, BS 2013

$$(1+\tau)^{2m-1}\psi_{2m}\left(\frac{\tau}{1+\tau}\right) - (1-\tau)^{2m-1}\psi_{2m}\left(\frac{\tau}{1-\tau}\right)$$
$$= \pi^{2m}\,\mathrm{rat}(\tau)$$

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THM LRR, BS 2013

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proof Collect residues of the integral

$$I_C = \frac{1}{2\pi i}\int_C \frac{\sin(\pi\tau z)}{\sin(\pi(1+\tau)z)\sin(\pi(1-\tau)z)}\frac{\mathrm{d}z}{z^{s+1}}.$$

C are appropriate circles around the origin such that $I_C \to 0$ as $\mathrm{radius}(C) \to \infty$. $\square$

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THM LRR, BS 2013

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$$= \pi^{2m}[z^{2m-1}]\frac{\sin(\tau z)}{\sin((1-\tau)z)\sin((1+\tau)z)}$$

- In terms of $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$,

$$\psi_{2m}|_{1-2m}(T^2 - 1) = 0,$$
$$\psi_{2m}|_{1-2m}(R^2 - 1) = \pi^{2m}\,\mathrm{rat}(\tau).$$

COR For any $\gamma \in \Gamma(2)$, $\quad \psi_{2m}|_{1-2m}(\gamma - 1) = \pi^{2m}\,\mathrm{rat}(\tau).$

Secant zeta function: Special values

THM LRR, BS 2013

For positive integers m, r,

$$\psi_{2m}(\sqrt{r}) \in \mathbb{Q} \cdot \pi^{2m}.$$

proof

- Any real quadratic irrational τ is fixed by some $\gamma \in \Gamma(2)$.
  This follows from Pell's equation.
- Combined with

$$\psi_{2m}|_{1-2m}(\gamma - 1) = \pi^{2m} \operatorname{rat}(\tau),$$

  it follows that $\psi_{2m}(\tau) \in \mathbb{Q}(\tau) \cdot \pi^{2m}$.
- Finally, use the fact that ψ_{2m} is even. $\square$

Eichler integrals

- F is an **Eichler integral** if $D^{k-1}F$ is modular of weight k. $D = q\frac{\mathrm{d}}{\mathrm{d}q}$

EG $$\sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n = \sum_{n=1}^{\infty} \frac{n^{2k-1} q^n}{1-q^n} \xrightarrow{\text{integrate}} \sum_{n=1}^{\infty} \frac{\sigma_{2k-1}(n)}{n^{2k-1}} q^n = \sum_{n=1}^{\infty} \frac{n^{1-2k} q^n}{1-q^n}$$

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- Eichler integrals are characterized by

$$F|_{2-k}(\gamma - 1) = \mathrm{poly}(\tau), \qquad \deg \mathrm{poly} \leqslant k-2.$$

- $\mathrm{poly}(\tau)$ is a **period polynomial** of the modular form f.
  The period polynomial encodes the critical L-values of f.

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- Eichler integrals are characterized by

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- $\mathrm{poly}(\tau)$ is a **period polynomial** of the modular form f.
  The period polynomial encodes the critical L-values of f.
- For a modular form $f(\tau) = \sum a(n) q^n$ of weight k, define

$$\tilde{f}(\tau) = \frac{(-1)^k \Gamma(k-1)}{(2\pi i)^{k-1}} \sum_{n=1}^{\infty} \frac{a(n)}{n^{k-1}} q^n.$$

- For the **Eisenstein series** G_{2k}, the period "polynomial" is

$$\tilde{G}_{2k}|_{2-2k}(S-1) = \frac{(2\pi i)^{2k}}{2k-1}\left[\sum_{s=0}^{k}\frac{B_{2s}}{(2s)!}\frac{B_{2k-2s}}{(2k-2s)!}X^{2s-1} + \frac{\zeta(2k-1)}{(2\pi i)^{2k-1}}(X^{2k-2}-1)\right].$$

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- In other words, $\sum \frac{\cot(\pi n \tau)}{n^{2k-1}}$ is an Eichler integral of G_{2k}.

EG

$$\cot(\pi\tau) = \frac{1}{\pi}\sum_{j\in\mathbb{Z}} \frac{1}{\tau+j}$$

$$\lim_{N\to\infty}\sum_{j=-N}^{N}$$

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EG

$$\cot(\pi\tau) = \frac{1}{\pi}\sum_{j\in\mathbb{Z}} \frac{1}{\tau+j}$$

$$\lim_{N\to\infty}\sum_{j=-N}^{N}$$

- $\sum \frac{\sec(\pi n\tau)}{n^{2k}}$ is an Eichler integral of an Eisenstein series as well.

EG

$$\sec\left(\frac{\pi\tau}{2}\right) = \frac{2}{\pi}\sum_{j\in\mathbb{Z}} \frac{\chi_{-4}(j)}{\tau+j}$$

Eisenstein series

- More generally, we have the Eisenstein series

$$E_k(\tau;\chi,\psi) = \sum_{m,n\in\mathbb{Z}}{}' \frac{\chi(m)\psi(n)}{(m\tau+n)^k},$$

  where χ and ψ are Dirichlet characters modulo L and M.
- We assume $\chi(-1)\psi(-1) = (-1)^k$. Otherwise, $E_k(\tau;\chi,\psi) = 0$.

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PROP Modular transformations: $\gamma = \left(\begin{smallmatrix} a & Mb \\ Lc & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$

- $E_k(\tau;\chi,\psi)|_k\gamma = \chi(d)\bar{\psi}(d)E_k(\tau;\chi,\psi)$
- $E_k(\tau;\chi,\psi)|_kS = \chi(-1)E_k(\tau;\psi,\chi)$

PROP If ψ is primitive, the L-function of $E(\tau) = E_k(\tau;\chi,\psi)$ is

$$L(E,s) = \mathrm{const}\cdot M^s L(\chi,s)L(\bar{\psi},1-k+s).$$

THM Berndt-S 2013

For $k \geqslant 3$, primitive χ, $\psi \neq 1$, and n such that $L|n$,

$$R^n = \left(\begin{smallmatrix} 1 & 0 \\ n & 1 \end{smallmatrix}\right)$$

$$\tilde{E}_k(X;\chi,\psi)|_{2-k}(1-R^n)$$

$$= \text{const} \sum_{s=0}^{k} \frac{B_{k-s,\bar{\chi}}}{(k-s)!L^{k-s}} \frac{B_{s,\bar{\psi}}}{s!M^s} X^{s-1}|_{2-k}(1-R^n).$$

$$\text{const} = -\chi(-1)G(\chi)G(\psi)\frac{(2\pi i)^k}{k-1}$$

- The **generalized Bernoulli numbers** appear because

$$L(1-n,\chi) = -B_{n,\chi}/n.$$

($n > 0$, primitive χ with $\chi(-1) = (-1)^n$)

- Note that $X^{s-1}|_{2-k}(1-R^n) = X^{s-1}(1-(nX+1)^{k-1-s})$.

Unimodular polynomials

DEF $p(x)$ is **unimodular** if all its zeros have absolute value 1.

EG

$$x^2 + \frac{6}{5}x + 1 = \left(x + \frac{3+4i}{5}\right)\left(x + \frac{3-4i}{5}\right)$$

- Kronecker: if $p(x) \in \mathbb{Z}[x]$ is monic and unimodular, then all of its roots are roots of unity. hence Mahler measure 1,

EG **Lehmer**

$$x^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$$

has only the two real roots 0.850, 1.176 off the unit circle.

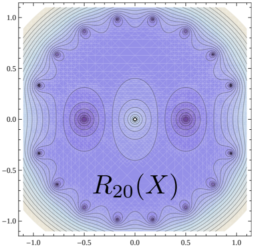
Lehmer's conjecture: $1.176\ldots$ is the smallest Mahler measure (greater than 1)

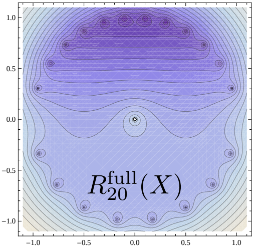
- Following Gun–Murty–Rath, the **Ramanujan polynomials** are

$$R_k(X) = \sum_{s=0}^{k} \frac{B_s}{s!} \frac{B_{k-s}}{(k-s)!} X^{s-1}.$$

THM Murty-Smyth-Wang '11: All nonreal zeros of $R_k(X)$ lie on the unit circle.
For $k \geqslant 2$, $R_{2k}(X)$ has exactly four real roots which approach $\pm 2^{\pm 1}$.

THM Lalín-Smyth '13: $R_{2k}(X) + \frac{\zeta(2k-1)}{(2\pi i)^{2k-1}}(X^{2k-2} - 1)$ is unimodular.





Unimodularity of period polynomials

THM Conrey-Farmer-Imamoglu 2012

For any Hecke cusp form (for $\mathrm{SL}_2(\mathbb{Z})$), the odd part of its period polynomial has

- trivial zeros at 0, ± 2, $\pm \frac{1}{2}$,
- and all remaining zeros lie on the unit circle.

THM El-Guindy–Raji 2013

For any Hecke eigenform (for $\mathrm{SL}_2(\mathbb{Z})$), the full period polynomial has all zeros on the unit circle.

Q What about higher level?

Generalized Ramanujan polynomials

- Consider the following **generalized Ramanujan polynomials**:

$$R_k(X;\chi,\psi) = \sum_{s=0}^{k} \frac{B_{s,\chi}}{s!} \frac{B_{k-s,\psi}}{(k-s)!} \left(\frac{X-1}{M}\right)^{k-s-1} (1-X^{s-1})$$

- Essentially, period polynomials: χ, ψ primitive, nonprincipal

$$R_k(LX+1;\chi,\psi) = \text{const} \cdot \tilde{E}_k(X;\bar{\chi},\bar{\psi})|_{2-k}(1-R^L)$$

PROP Berndt-S 2013

- For even $k>1$,

$$R_k(X;1,1) = \sum_{s=0}^{k} \frac{B_s}{s!} \frac{B_{k-s}}{(k-s)!} X^{s-1}.$$

- $R_k(X;\chi,\psi)$ is self-inversive.

CONJ If χ,ψ are nonprincipal real, then $R_k(X;\chi,\psi)$ is unimodular.

Special values of trigonometric Dirichlet series

EG Ramanujan

$$\sum_{n=0}^{\infty} \frac{\tanh((2n+1)\pi/2)}{(2n+1)^3} = \frac{\pi^3}{32}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\operatorname{csch}(\pi n)}{n^3} = \frac{\pi^3}{360}$$

EG Berndt 1976-78

$$\sum_{n=1}^{\infty} \frac{\cot(\pi n\sqrt{7})}{n^3} = -\frac{\sqrt{7}}{20}\pi^3, \qquad \sum_{n=0}^{\infty} \frac{\tan(\pi(2n+1)\sqrt{5})}{(2n+1)^5} = \frac{23\pi^5}{3456\sqrt{5}}$$

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EG Komori-Matsumoto-Tsumura 2013

$$\sum_{n=1}^{\infty} \frac{\cot^2(\pi n\zeta_3)}{n^4} = -\frac{31}{2835}\pi^4, \qquad \sum_{n=1}^{\infty} \frac{\csc^2(\pi n\zeta_3)}{n^4} = \frac{1}{5670}\pi^4$$

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THM S 2014

Let $r \in \mathbb{Q}$, and let $a, b, s \in \mathbb{Z}$ be such that $s \geqslant \max(a, b, 1) + 1$, s and b have the same parity, and $a + b \geqslant 0$. Then,

$$\sum_{n=1}^{\infty} \frac{\operatorname{trig}^{a,b}(\pi n\sqrt{r})}{n^s} \in (\pi\sqrt{r})^s\mathbb{Q}, \qquad \operatorname{trig}^{a,b} = \sec^a \csc^b.$$

Special values of trigonometric Dirichlet series

EG
S 2014

$$\sum_{n=1}^{\infty} \frac{\sec^2(\pi n\sqrt{5})}{n^4} = \frac{14}{135}\pi^4$$

$$\sum_{n=1}^{\infty} \frac{\cot^2(\pi n\sqrt{5})}{n^4} = \frac{13}{945}\pi^4$$

$$\sum_{n=1}^{\infty} \frac{\csc^2(\pi n\sqrt{11})}{n^4} = \frac{8}{385}\pi^4$$

$$\sum_{n=1}^{\infty} \frac{\sec^3(\pi n\sqrt{2})}{n^4} = -\frac{2483}{5220}\pi^4$$

$$\sum_{n=1}^{\infty} \frac{\tan^3(\pi n\sqrt{6})}{n^5} = \frac{35,159}{17,820\sqrt{6}}\pi^4$$

THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`

B. Berndt, A. Straub
On a secant Dirichlet series and Eichler integrals of Eisenstein series
Preprint, 2013

A. Straub
Special values of trigonometric Dirichlet series and Eichler integrals
In preparation, 2014

THM Berndt-S 2013

For $\alpha \in \mathcal{H}$, such that $R_k(\alpha; \bar{\chi}, 1) = 0$ and $\alpha^{k-2} \neq 1$,
($k \geqslant 3$, χ primitive, $\chi(-1) = (-1)^k$)

$$L(k-1, \chi) = \frac{k-1}{2\pi i(1-\alpha^{k-2})} \left[\tilde{E}_k \left(\frac{\alpha - 1}{L}; \chi, 1 \right) - \alpha^{k-2} \tilde{E}_k \left(\frac{1 - 1/\alpha}{L}; \chi, 1 \right) \right]$$

$$= \frac{2}{1-\alpha^{k-2}} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{k-1}} \left[\frac{1}{1 - e^{2\pi i n(1-\alpha)/L}} - \frac{\alpha^{k-2}}{1 - e^{2\pi i n(1/\alpha - 1)/L}} \right].$$

THM Berndt-S 2013

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$$= \frac{2}{1-\alpha^{k-2}} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{k-1}} \left[\frac{1}{1 - e^{2\pi i n(1-\alpha)/L}} - \frac{\alpha^{k-2}}{1 - e^{2\pi i n(1/\alpha - 1)/L}} \right].$$

THM Gun–Murty–Rath 2011

As $\beta \in \mathcal{H}$, $\beta^{2k-2} \neq 1$, ranges over algebraic numbers, the values

$$\frac{1}{\pi} \left[\tilde{E}_{2k}(\beta; 1, 1) - \beta^{2k-2} \tilde{E}_{2k}(-1/\beta; 1, 1) \right]$$

contain at most one algebraic number.