Properties and applications of Apéry-like numbers

Number Theory Seminar
University of Illinois at Urbana-Champaign

Armin Straub

April 3, 2014

University of Illinois at Urbana–Champaign

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

$$1, 5, 73, 1445, 33001, 819005, 21460825, \ldots$$

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2 + 17n + 5)u_n - n^3 u_{n-1}.$$

Apéry numbers and the irrationality of $\zeta(3)$

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2+17n+5)u_n - n^3 u_{n-1}.$$

THM (Apéry '78) $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.

proof The same recurrence is satisfied by the "near"-integers

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right).$$

Then, $\frac{B(n)}{A(n)} \to \zeta(3)$. But too fast for $\zeta(3)$ to be rational. □

Apéry-like numbers

- Recurrence for Apéry numbers is the case $(a, b, c) = (17, 5, 1)$ of

$$(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - cn^3 u_{n-1}.$$

Q Are there other tuples (a, b, c) for which the solution defined by $u_{-1} = 0$, $u_0 = 1$ is integral?

Apéry-like numbers

- Recurrence for Apéry numbers is the case $(a, b, c) = (17, 5, 1)$ of

$$(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - cn^3 u_{n-1}.$$

Q Are there other tuples (a, b, c) for which the solution defined by $u_{-1} = 0$, $u_0 = 1$ is integral?

- Essentially, only 14 tuples (a, b, c) found. (Almkvist–Zudilin)
  - 4 hypergeometric and 4 Legendrian solutions
  - 6 sporadic solutions

Apéry-like numbers

- Recurrence for Apéry numbers is the case $(a,b,c)=(17,5,1)$ of

$$(n+1)^3 u_{n+1} = (2n+1)(an^2+an+b)u_n - cn^3 u_{n-1}.$$

Q Are there other tuples (a,b,c) for which the solution defined by $u_{-1}=0$, $u_0=1$ is integral?

- Essentially, only 14 tuples (a,b,c) found. (Almkvist–Zudilin)
  - 4 hypergeometric and 4 Legendrian solutions
  - 6 sporadic solutions

- Similar (and intertwined) story for:
  - $(n+1)^2 u_{n+1} = (an^2+an+b)u_n - cn^2 u_{n-1}$ (Beukers, Zagier)
  - $(n+1)^3 u_{n+1} = (2n+1)(an^2+an+b)u_n - n(cn^2+d)u_{n-1}$ (Cooper)

Apéry-like numbers

- Hypergeometric and Legendrian solutions have generating functions

$$ {}_3F_2\left(\begin{matrix}\frac{1}{2},\alpha,1-\alpha\\ 1,1\end{matrix}\middle|4C_\alpha z\right), \qquad \frac{1}{1-C_\alpha z}\,{}_2F_1\left(\begin{matrix}\alpha,1-\alpha\\ 1\end{matrix}\middle|\frac{-C_\alpha z}{1-C_\alpha z}\right)^2, $$

  with $\alpha = \frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{6}$ and $C_\alpha = 2^4, 3^3, 2^6, 2^4\cdot 3^3$.

- The six sporadic solutions are:

(a,b,c)	$A(n)$
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k}\binom{n}{3k}\binom{n+k}{n}\frac{(3k)!}{k!^3}$
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3\left(\binom{4n-5k-1}{3n}+\binom{4n-5k}{3n}\right)$
$(10,4,64)$	$\sum_k \binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}$
$(12,4,16)$	$\sum_k \binom{n}{k}^2\binom{2k}{n}^2$
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2\binom{n}{l}\binom{k}{l}\binom{k+l}{n}$
$(17,5,1)$	$\sum_k \binom{n}{k}^2\binom{n+k}{n}^2$

Modularity of Apéry-like numbers

- The **Apéry numbers** $1, 5, 73, 1145, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n \geqslant 0} A(n) \underbrace{\left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)}\right)^{12n}}_{\text{modular function}}.$$

- The **Apéry numbers** $1, 5, 73, 1145, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n \geqslant 0} A(n) \underbrace{\left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)}\right)^{12n}}_{\text{modular function}}.$$

FACT Not at all evidently, such a **modular parametrization** exists for all known Apéry-like numbers!

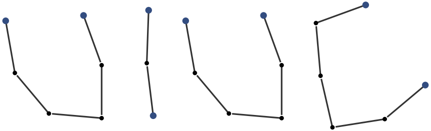
- Context:

$f(\tau)$	modular form of weight k
$x(\tau)$	modular function
$y(x)$	such that $y(x(\tau)) = f(\tau)$

Then $y(x)$ satisfies a linear differential equation of order $k+1$.

EXAMPLE I

Short random walks



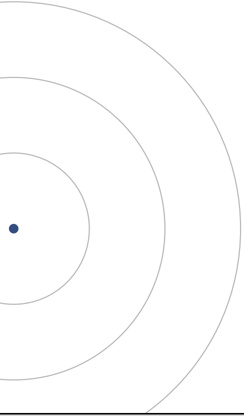
Based on joint work with:



Jon Borwein James Wan Wadim Zudilin
(University of Newcastle, Australia)

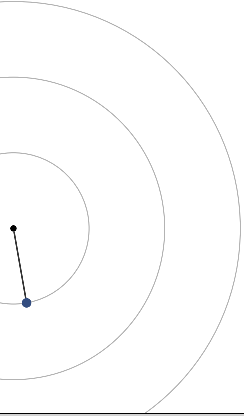
Example I: Random walks

n steps in the plane
(length 1, random direction)



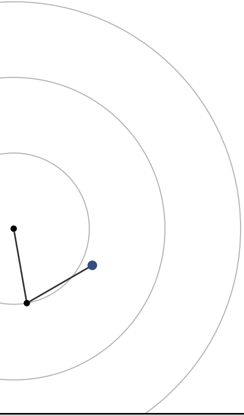
Example I: Random walks

n steps in the plane
(length 1, random direction)



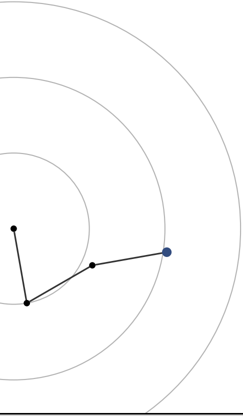
Example I: Random walks

n steps in the plane
(length 1, random direction)



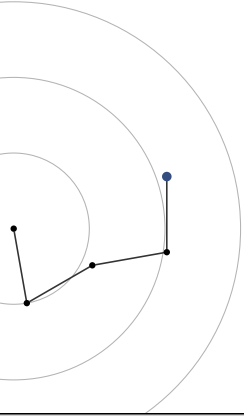
Example I: Random walks

n steps in the plane
(length 1, random direction)



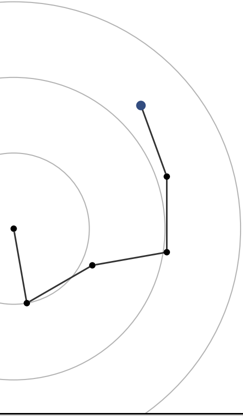
Example I: Random walks

n steps in the plane
(length 1, random direction)



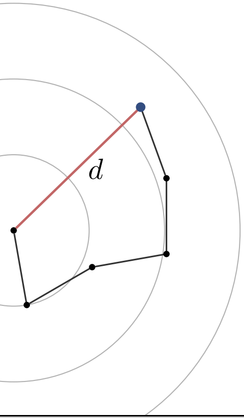
Example I: Random walks

n steps in the plane
(length 1, random direction)



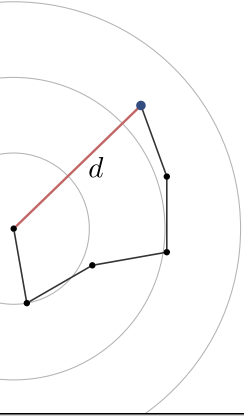
Example I: Random walks

n steps in the plane
(length 1, random direction)

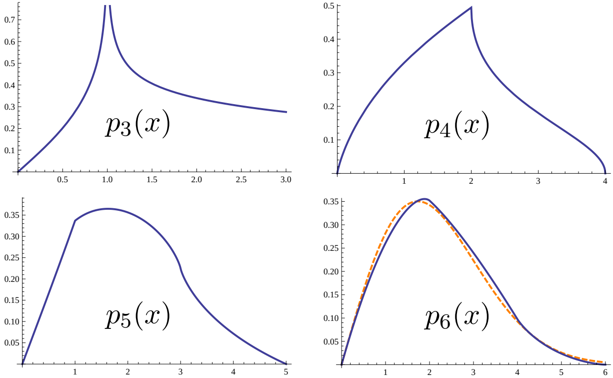


Example I: Random walks

n steps in the plane
(length 1, random direction)

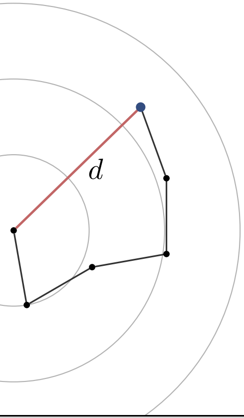


- $p_n(x)$ — probability density of distance traveled

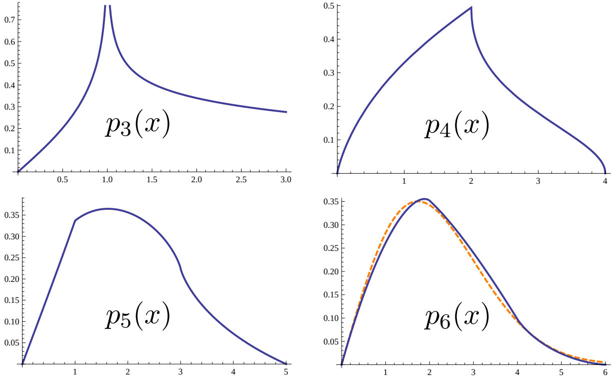


Example I: Random walks

n steps in the plane
(length 1, random direction)



- $p_n(x)$ — probability density of distance traveled



- $W_n(s) = \int_0^\infty x^s p_n(x)\,\mathrm{d}x$ — probability moments

$$W_2(1) = \frac{4}{\pi}, \qquad W_3(1) = \frac{3}{16}\frac{2^{1/3}}{\pi^4}\Gamma^6\left(\frac{1}{3}\right) + \frac{27}{4}\frac{2^{2/3}}{\pi^4}\Gamma^6\left(\frac{2}{3}\right)$$

classical — Borwein–Nuyens–S–Wan, 2010

Example I: Random walks

- The probability moments

$$W_n(s) = \int_0^\infty x^s p_n(x)\,\mathrm{d}x$$

include the Apéry-like numbers

$$W_3(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j},$$

$$W_4(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} \binom{2(k-j)}{k-j}.$$

Example I: Random walks

- The probability moments

$$W_n(s) = \int_0^\infty x^s p_n(x)\,\mathrm{d}x$$

include the Apéry-like numbers

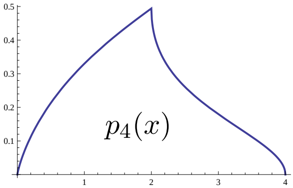
$$W_3(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j},$$

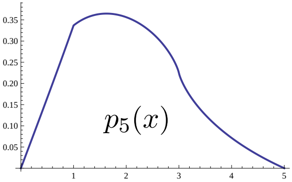
$$W_4(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} \binom{2(k-j)}{k-j}.$$

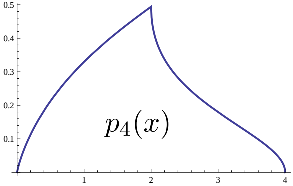
THM Borwein-Nuyens-S-Wan 2010

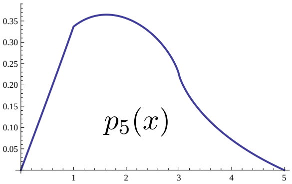
$$W_n(2k) = \sum_{a_1+\cdots+a_n=k} \binom{k}{a_1,\ldots,a_n}^2$$

Example I: Random walks







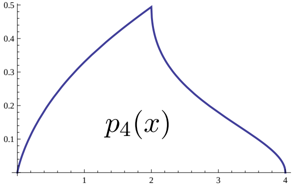


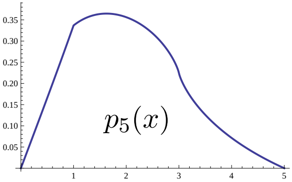
THM Borwein-S-Wan-Zudilin 2011

For $\tau = -1/2 + iy$ and $y > 0$:

$$p_4\left(\underbrace{8i\left(\frac{\eta(2\tau)\eta(6\tau)}{\eta(\tau)\eta(3\tau)}\right)^3}_{\text{modular function}}\right) = \frac{6(2\tau+1)}{\pi}\underbrace{\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)}_{\text{modular form}}$$

Example I: Random walks





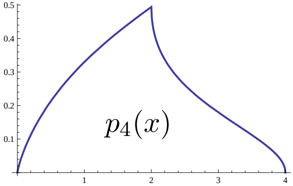
$$p_5'(0) = p_4(1) \approx 0.32993$$

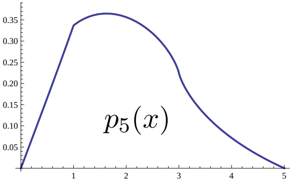
THM Borwein-S-Wan-Zudilin 2011

For $\tau = -1/2 + iy$ and $y > 0$:

$$p_4\left(\underbrace{8i\left(\frac{\eta(2\tau)\eta(6\tau)}{\eta(\tau)\eta(3\tau)}\right)^3}_{\text{modular function}}\right) = \frac{6(2\tau+1)}{\pi}\underbrace{\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)}_{\text{modular form}}$$

Example I: Random walks





$$p_5'(0) = p_4(1) \approx 0.32993$$

THM Borwein-S-Wan-Zudilin 2011

For $\tau = -1/2 + iy$ and $y > 0$:

$$p_4\left(\underbrace{8i\left(\frac{\eta(2\tau)\eta(6\tau)}{\eta(\tau)\eta(3\tau)}\right)^3}_{\text{modular function}}\right) = \frac{6(2\tau+1)}{\pi}\underbrace{\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)}_{\text{modular form}}$$

- When $\tau = -\frac{1}{2} + \frac{1}{6}\sqrt{-15}$, one obtains $p_4(1)$ as an eta-product.
- Modular equations and Chowla–Selberg lead to:

$$p_4(1) = \frac{\sqrt{5}}{40\pi^4}\,\Gamma(\tfrac{1}{15})\Gamma(\tfrac{2}{15})\Gamma(\tfrac{4}{15})\Gamma(\tfrac{8}{15}) \approx 0.32993$$

EXAMPLE II

Binomial congruences

$$\binom{2p-1}{p-1} \equiv 1 \mod p^3$$

Includes joint work with:

Robert Osburn
(University of Dublin)

Brundaban Sahu
(NISER, India)

Example II: Binomial congruences

John Wilson (1773, Lagrange): $(p-1)! \equiv -1 \mod p$

Charles Babbage (1819): $\binom{2p-1}{p-1} \equiv 1 \mod p^2$

Joseph Wolstenholme (1862): $\binom{2p-1}{p-1} \equiv 1 \mod p^3$

James W.L. Glaisher (1900): $\binom{mp-1}{p-1} \equiv 1 \mod p^3$

Wilhelm Ljunggren (1952): $\binom{ap}{bp} \equiv \binom{a}{b} \mod p^3$

Example II: Binomial congruences

John Wilson (1773, Lagrange): $$(p-1)! \equiv -1 \mod p$$

Charles Babbage (1819): $$\binom{2p-1}{p-1} \equiv 1 \mod p^2$$

Joseph Wolstenholme (1862): $$\binom{2p-1}{p-1} \equiv 1 \mod p^3$$

James W.L. Glaisher (1900): $$\binom{mp-1}{p-1} \equiv 1 \mod p^3$$

Wilhelm Ljunggren (1952): $$\binom{ap}{bp} \equiv \binom{a}{b} \mod p^3$$

THM Clark, Andrews '95–'99

$$\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^{p^2}}, \qquad \binom{ap}{bp}_q \equiv q^{(a-b)b\binom{p}{2}} \binom{a}{b}_{q^p} \mod [p]_q^2$$

Example II: Binomial congruences

John Wilson (1773, Lagrange): $$(p-1)! \equiv -1 \mod p$$

Charles Babbage (1819): $$\binom{2p-1}{p-1} \equiv 1 \mod p^2$$

Joseph Wolstenholme (1862): $$\binom{2p-1}{p-1} \equiv 1 \mod p^3$$

James W.L. Glaisher (1900): $$\binom{mp-1}{p-1} \equiv 1 \mod p^3$$

Wilhelm Ljunggren (1952): $$\binom{ap}{bp} \equiv \binom{a}{b} \mod p^3$$

THM
S 2011
$p \geqslant 5$

$$\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^{p^2}} - \binom{a}{b+1}\binom{b+1}{2}\frac{p^2-1}{12}(q^p-1)^2 \mod [p]_q^3$$

Example II: Binomial congruences

- Wolstenholme's congruence is the $m = 1$ case of:
  The sequence $A(n) = \binom{2n}{n}$ satisfies the **supercongruence** $(p \geqslant 5)$

$$A(mp) \equiv A(m) \mod p^3.$$

Example II: Binomial congruences

- Wolstenholme's congruence is the $m = 1$ case of:
  The sequence $A(n) = \binom{2n}{n}$ satisfies the **supercongruence** $(p \geqslant 5)$

$$A(mp) \equiv A(m) \mod p^3.$$

- The same congruence is satisfied by the **Apéry numbers**

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2.$$

Example II: Binomial congruences

- Wolstenholme's congruence is the $m = 1$ case of:
  The sequence $A(n) = \binom{2n}{n}$ satisfies the **supercongruence** $(p \geqslant 5)$

$$A(mp) \equiv A(m) \mod p^3.$$

- The same congruence is satisfied by the **Apéry numbers**

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2.$$

- In fact,

$$A(mp^r) \equiv A(mp^{r-1}) \mod p^{3r}.$$

Example II: Binomial congruences

- Wolstenholme's congruence is the $m = 1$ case of:
  The sequence $A(n) = \binom{2n}{n}$ satisfies the **supercongruence** $(p \geqslant 5)$

$$A(mp) \equiv A(m) \mod p^3.$$

- The same congruence is satisfied by the **Apéry numbers**

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2.$$

- In fact,

$$A(mp^r) \equiv A(mp^{r-1}) \mod p^{3r}.$$

Q How does the q-side of supercongruences for Apéry-like numbers look like?

Example II: Binomial congruences

- Conjecturally, supercongruences like

$$A(mp^r) \equiv A(mp^{r-1}) \mod p^{3r}$$

hold for all Apéry-like numbers. Osburn–Sahu '09

- Current state of affairs for the six sporadic sequences from earlier:

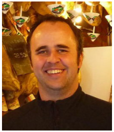
(a,b,c)	$A(n)$	
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$	open!!
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n} + \binom{4n-5k}{3n} \right)$	Osburn–Sahu–S '13
$(10,4,64)$	$\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$	Osburn–Sahu '11
$(12,4,16)$	$\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$	Osburn–Sahu–S '13
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$	open
$(17,5,1)$	$\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$	Beukers, Coster '87-'88

EXAMPLE III

Ramanujan-type series for $1/\pi$

$$\frac{2}{\pi} = 1 - 5\left(\frac{1}{2}\right)^3 + 9\left(\frac{1.3}{2.4}\right)^3 - 13\left(\frac{1.3.5}{2.4.6}\right)^3 + \ldots$$

Based on joint work with:



Mathew Rogers
(University of Montreal)

Example III: Series for $1/\pi$

$$\frac{4}{\pi} = 1 + \frac{7}{4}\left(\frac{1}{2}\right)^3 + \frac{13}{4^2}\left(\frac{1.3}{2.4}\right)^3 + \frac{19}{4^3}\left(\frac{1.3.5}{2.4.6}\right)^3 + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3}(6n+1)\frac{1}{4^n}$$

$$\frac{8}{\pi} = \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3}(42n+5)\frac{1}{2^{6n}}$$

- Starred in High School Musical, a 2006 Disney production

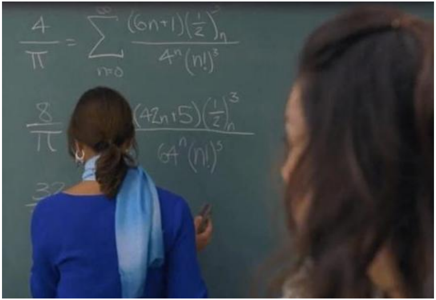
Srinivasa Ramanujan
Modular equations and approximations to π
Quart. J. Math., Vol. 45, p. 350–372, 1914

Example III: Series for $1/\pi$

$$\frac{4}{\pi} = 1 + \frac{7}{4}\left(\frac{1}{2}\right)^3 + \frac{13}{4^2}\left(\frac{1.3}{2.4}\right)^3 + \frac{19}{4^3}\left(\frac{1.3.5}{2.4.6}\right)^3 + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3}(6n+1)\frac{1}{4^n}$$

$$\frac{\mathbf{16}}{\pi} = \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3}(42n+5)\frac{1}{2^{6n}}$$



- Starred in High School Musical, a 2006 Disney production

Srinivasa Ramanujan
Modular equations and approximations to π
Quart. J. Math., Vol. 45, p. 350–372, 1914

Example III: Series for $1/\pi$

EG Gosper 1985

$$\frac{1}{\pi}=\frac{2\sqrt{2}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!}{n!^4}\frac{1103+26390n}{396^{4n}}$$

EG Chudnovsky's 1988

$$\frac{1}{\pi}=12\sum_{n=0}^{\infty}\frac{(-1)^n(6n)!}{(3n)!n!^3}\frac{13591409+545140134n}{640320^{3n+3/2}}$$

Example III: Series for $1/\pi$

EG Gosper 1985

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{n!^4} \frac{1103 + 26390n}{396^{4n}}$$

EG Chudnovsky's 1988

$$\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} \frac{(-1)^n (6n)!}{(3n)! n!^3} \frac{13591409 + 545140134n}{640320^{3n+3/2}}$$

- Sun offered a \$520 bounty for a proof the following series:

THM Rogers-S 2012

$$\frac{520}{\pi} = \sum_{n=0}^{\infty} \frac{1054n + 233}{480^n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} (-1)^k 8^{2k-n}$$

- $\binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} = \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^3$ is an Apery-like sequence.

Example III: Series for $1/\pi$

- Suppose we have a sequence a_n with **modular parametrization**

$$\sum_{n=0}^{\infty} a_n \underbrace{x(\tau)^n}_{\text{modular function}} = \underbrace{f(\tau)}_{\text{modular form}} .$$

- Then:

$$\sum_{n=0}^{\infty} a_n (A + Bn) x(\tau)^n = A f(\tau) + B \frac{x(\tau)}{x'(\tau)} f'(\tau)$$

$$\sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3} (42n + 5) \frac{1}{2^{6n}} = \frac{16}{\pi}$$

- Suppose we have a sequence a_n with **modular parametrization**

$$\sum_{n=0}^{\infty} a_n \underbrace{x(\tau)^n}_{\text{modular function}} = \underbrace{f(\tau)}_{\text{modular form}} .$$

- Then:

$$\sum_{n=0}^{\infty} a_n (A + Bn) x(\tau)^n = A f(\tau) + B \frac{x(\tau)}{x'(\tau)} f'(\tau)$$

$$\sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3} (42n+5) \frac{1}{2^{6n}} = \frac{16}{\pi}$$

FACT

- For $\tau \in \mathbb{Q}(\sqrt{-d})$, $x(\tau)$ is an algebraic number.
- $f'(\tau)$ is a **quasimodular** form.
- Prototypical $E_2(\tau)$ satisfies $\tau^{-2} E_2\left(-\frac{1}{\tau}\right) - E_2(\tau) = \frac{6}{\pi i \tau}$.

- These are the main ingredients for series for $1/\pi$. Mix and stir.

EXAMPLE IV

Positivity of rational functions

$$\frac{1}{1-(x+y+z+w)+2(yzw+xzw+xyw+xyz)+4xyzw}$$

Based on joint work with:

Wadim Zudilin
(University of Newcastle)

Example IV: Positivity

- A rational function

$$F(x_1, \ldots, x_d) = \sum_{n_1,\ldots,n_d \geqslant 0} a_{n_1,\ldots,n_d} x_1^{n_1} \cdots x_d^{n_d}$$

is **positive** if $a_{n_1,\ldots,n_d} > 0$ for all indices.

EG The following rational functions are positive.

$$S(x,y,z) = \frac{1}{1-(x+y+z)+\frac{3}{4}(xy+yz+zx)}$$

Szegő '33
Kaluza '33
Askey–Gasper '72
S '08

$$A(x,y,z) = \frac{1}{1-(x+y+z)+4xyz}$$

Askey–Gasper '77
Koornwinder '78
Ismail–Tamhankar '79
Gillis–Reznick–Zeilberger '83

- Both functions are on the boundary of positivity.

Example IV: Positivity

- WZ shows that the diagonal terms a_n of $A(x, y, z)$ satisfy

$$(n+1)^2 a_{n+1} = (7n^2 + 7n + 2) a_n + 8n^2 a_{n-1}.$$

  This proves that they equal the **Franel numbers**

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^3.$$

- Using the modular parametrization of the associated Calabi–Yau differential equation, we have

$$\sum_{n=0}^{\infty} a_n z^n = \frac{1}{1-2z} \, {}_2F_1 \left(\begin{matrix} \frac{1}{3}, \frac{2}{3} \\ 1 \end{matrix} \middle| \frac{27z^2}{(1-2z)^3} \right).$$

Example IV: Positivity

- Long-standing Lewy–Askey problem asks for positivity of

$$\frac{1}{1-(x+y+z+w)+\frac{2}{3}(xy+xz+xw+yz+yw+zw)}.$$

Recent proof of non-negativity by Scott and Sokal, 2013

- Implied by conjectured positivity of the Kauers–Zeilberger function

$$\frac{1}{1-(x+y+z+w)+2(yzw+xzw+xyw+xyz)+4xyzw}.$$

PROP S-Zudilin 2013 The Kauers–Zeilberger function has diagonal coefficients

$$d_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}^2.$$

Example IV: Positivity

- Under what condition(s) is the positivity of a rational function

$$h(x_1, \ldots, x_d) = \frac{1}{\sum_{k=0}^{d} c_k e_k(x_1, \ldots, x_d)}$$

implied by the positivity of its diagonal?

EG $\frac{1}{1+x+y}$ has positive diagonal coefficients but is not positive.

- Is the positivity of $h(x_1, \ldots, x_{d-1}, 0)$ a sufficient condition?

Example IV: Positivity

- Under what condition(s) is the positivity of a rational function

$$h(x_1, \ldots, x_d) = \frac{1}{\sum_{k=0}^{d} c_k e_k(x_1, \ldots, x_d)}$$

implied by the positivity of its diagonal?

EG $\frac{1}{1+x+y}$ has positive diagonal coefficients but is not positive.

- Is the positivity of $h(x_1, \ldots, x_{d-1}, 0)$ a sufficient condition?

THM S-Zudilin 2013

$$h(x, y) = \frac{1}{1 + c_1(x + y) + c_2 xy}$$

is positive iff $h(x, 0)$ and the diagonal of $h(x, y)$ are positive.

OUTLOOK

Multivariate Apéry numbers

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

$$1, 5, 73, 1445, 33001, 819005, 21460825, \ldots$$

Apéry numbers as diagonals

- Given a series

$$F(x_1, \ldots, x_d) = \sum_{n_1, \ldots, n_d \geqslant 0} a(n_1, \ldots, n_d) x_1^{n_1} \cdots x_d^{n_d},$$

its **diagonal coefficients** are the coefficients $a(n, \ldots, n)$.

Apéry numbers as diagonals

- Given a series

$$F(x_1, \ldots, x_d) = \sum_{n_1, \ldots, n_d \geqslant 0} a(n_1, \ldots, n_d) x_1^{n_1} \cdots x_d^{n_d},$$

its **diagonal coefficients** are the coefficients $a(n, \ldots, n)$.

THM S 2013 The Apéry numbers are the diagonal coefficients of

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4}.$$

Apéry numbers as diagonals

- Given a series

$$F(x_1, \ldots, x_d) = \sum_{n_1,\ldots,n_d \geqslant 0} a(n_1, \ldots, n_d) x_1^{n_1} \cdots x_d^{n_d},$$

its **diagonal coefficients** are the coefficients $a(n, \ldots, n)$.

THM
S 2013
The Apéry numbers are the diagonal coefficients of

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4}.$$

- Previously known: they are also the diagonal of Christol, '84

$$\frac{1}{(1 - x_1)\left[(1 - x_2)(1 - x_3)(1 - x_4)(1 - x_5) - x_1 x_2 x_3\right]}.$$

- Such identities are routine to prove, but much harder to discover.

Apéry numbers as diagonals

- Given a series

$$F(x_1, \ldots, x_d) = \sum_{n_1, \ldots, n_d \geqslant 0} a(n_1, \ldots, n_d) x_1^{n_1} \cdots x_d^{n_d},$$

its **diagonal coefficients** are the coefficients $a(n, \ldots, n)$.

THM **S 2013** The Apéry numbers are the diagonal coefficients of

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4}.$$

- Univariate generating function:

$$\sum_{n \geq 0} A(n) x^n = \frac{17 - x - z}{4\sqrt{2}(1 + x + z)^{3/2}} \, {}_3F_2 \left(\left. \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} \right| - \frac{1024x}{(1 - x + z)^4} \right),$$

where $z = \sqrt{1 - 34x + x^2}$.

- Denote with $A(\boldsymbol{n}) = A(n_1, n_2, n_3, n_4)$ the coefficients of

$$\frac{1}{(1-x_1-x_2)(1-x_3-x_4) - x_1x_2x_3x_4}.$$

THM S 2013 Let $\boldsymbol{n} = (n_1, n_2, n_3, n_4) \in \mathbb{Z}_{\geqslant 0}^4$. For primes $p \geqslant 5$,

$$A(\boldsymbol{n}p^r) \equiv A(\boldsymbol{n}p^{r-1}) \mod p^{3r}.$$

Multivariable supercongruences

- Denote with $A(\boldsymbol{n}) = A(n_1, n_2, n_3, n_4)$ the coefficients of

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4}.$$

THM S 2013: Let $\boldsymbol{n} = (n_1, n_2, n_3, n_4) \in \mathbb{Z}_{\geqslant 0}^4$. For primes $p \geqslant 5$,

$$A(\boldsymbol{n} p^r) \equiv A(\boldsymbol{n} p^{r-1}) \mod p^{3r}.$$

- Note that if

$$\sum_{n \geqslant 0} a(n) x^n = F(x), \qquad \zeta_p = e^{2\pi i/p}$$

then

$$\sum_{n \geqslant 0} a(pn) x^{pn} = \frac{1}{p} \sum_{k=0}^{p-1} F(\zeta_p^k x).$$

Just some of the many open problems

- Supercongruences for all Apéry-like numbers
  - proof for all of them
  - uniform explanation
  - multivariable extensions
- Apéry-like numbers as diagonals
  - find minimal rational functions
  - extend supercongruences
  - any structure?
- Many further questions remain.
  - is the known list complete?
  - higher-order analogs, Calabi–Yau DEs
  - reason for modularity
  - modular supercongruences — Beukers, Ahlgren–Ono

$$A\left(\frac{p-1}{2}\right) \equiv a(p) \pmod{p^2}, \qquad \sum_{n=1}^{\infty} a_n q^n = \eta^4(2\tau)\eta^4(4\tau)$$

  - q-analogs
  - ...

THANK YOU!

Slides for this talk will be available from my website:
http://arminstraub.com/talks

A. Straub
Multivariate Apéry numbers and supercongruences of rational functions
Preprint, 2014

R. Osburn, B. Sahu, A. Straub
Supercongruences for sporadic sequences
Preprint, 2013

A. Straub, W. Zudilin
Positivity of rational functions and their diagonals
Preprint, 2013

M. Rogers, A. Straub
A solution of Sun's $520 challenge concerning $520/\pi$
International Journal of Number Theory, Vol. 9, Nr. 5, 2013, p. 1273-1288

J. Borwein, A. Straub, J. Wan, W. Zudilin (appendix by D. Zagier)
Densities of short uniform random walks
Canadian Journal of Mathematics, Vol. 64, Nr. 5, 2012, p. 961-990

A. Straub
A q*-analog of Ljunggren's binomial congruence*
DMTCS Proceedings: FPSAC 2011, p. 897-902

A. Straub
Positivity of Szegö's rational function
Advances in Applied Mathematics, Vol. 41, Issue 2, Aug 2008, p. 255-264