On a secant Dirichlet series and Eichler integrals of Eisenstein series

Oberseminar Zahlentheorie
Universität zu Köln

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PART I

A secant Dirichlet series

$$\psi_s(\tau) = \sum_{n=1}^{\infty} \frac{\sec(\pi n \tau)}{n^s}$$

- Lalín, Rodrigue and Rogers introduce and study

$$\psi_s(\tau) = \sum_{n=1}^{\infty} \frac{\sec(\pi n \tau)}{n^s}.$$

- Clearly, $\psi_s(0) = \zeta(s)$. In particular, $\psi_2(0) = \frac{\pi^2}{6}$.

EG
LRR '13

$$\psi_2(\sqrt{2}) = -\frac{\pi^2}{3}, \qquad \psi_2(\sqrt{6}) = \frac{2\pi^2}{3}$$

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CONJ
LRR '13
For positive integers m, r,

$$\psi_{2m}(\sqrt{r}) \in \mathbb{Q} \cdot \pi^{2m}.$$

Secant zeta function: Motivation

- Euler's identity:

$$\sum_{n=1}^{\infty} \frac{1}{n^{2m}} = -\frac{1}{2}(2\pi i)^{2m} \frac{B_{2m}}{(2m)!}$$

- Half of the Clausen and Glaisher functions reduce, e.g.,

$$\sum_{n=1}^{\infty} \frac{\cos(n\tau)}{n^{2m}} = \mathrm{poly}_m(\tau), \qquad \mathrm{poly}_1(\tau) = \frac{\tau^2}{4} - \frac{\pi\tau}{2} + \frac{\pi^2}{6}.$$

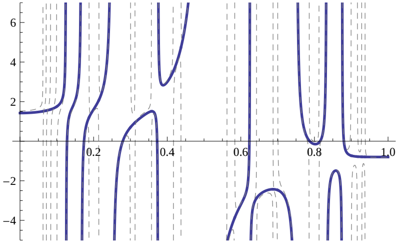
- Ramanujan investigated trigonometric Dirichlet series of similar type. From his first letter to Hardy:

$$\sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^7} = \frac{19\pi^7}{56700}$$

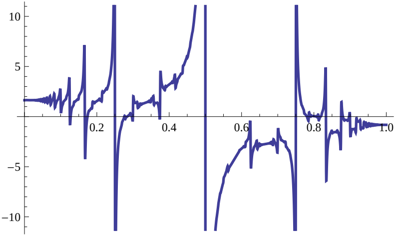
In fact, this was already included in a general formula by Lerch.

Secant zeta function: Convergence

- $\psi_s(\tau) = \sum \frac{\sec(\pi n\tau)}{n^s}$ has singularity at rationals with even denominator



$\psi_2(\tau)$ truncated to 4 and 8 terms



$\mathrm{Re}\ \psi_2(\tau + \varepsilon i)$ with $\varepsilon = 1/1000$

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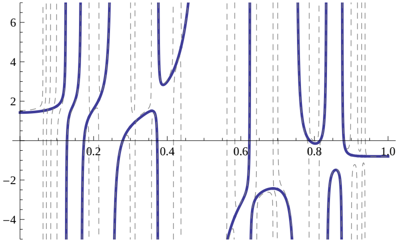

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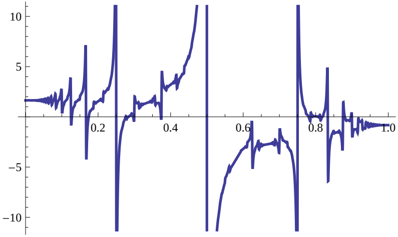

Re $\psi_2(\tau + \varepsilon i)$ with $\varepsilon = 1/1000$

THM Lalín–Rodrigue–Rogers 2013

The series $\psi_s(\tau) = \sum \frac{\sec(\pi n \tau)}{n^s}$ converges absolutely if

1. $\tau = p/q$ with q odd and $s > 1$,
2. τ is algebraic irrational and $s \geqslant 2$.

- Proof uses Thue–Siegel–Roth, as well as a result of Worley when $s = 2$ and τ is irrational

Secant zeta function: Functional equation

- Obviously, $\psi_s(\tau) = \sum \frac{\sec(\pi n \tau)}{n^s}$ satisfies $\psi_s(\tau + 2) = \psi_s(\tau)$.

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$$(1+\tau)^{2m-1}\psi_{2m}\left(\frac{\tau}{1+\tau}\right) - (1-\tau)^{2m-1}\psi_{2m}\left(\frac{\tau}{1-\tau}\right)$$
$$= \pi^{2m}\,\mathrm{rat}(\tau)$$

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proof Collect residues of the integral

$$I_C = \frac{1}{2\pi i}\int_C \frac{\sin(\pi\tau z)}{\sin(\pi(1+\tau)z)\sin(\pi(1-\tau)z)}\frac{\mathrm{d}z}{z^{s+1}}.$$

C are appropriate circles around the origin such that $I_C \to 0$ as $\operatorname{radius}(C) \to \infty$. $\square$

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$$= \pi^{2m}[z^{2m-1}]\frac{\sin(\tau z)}{\sin((1-\tau)z)\sin((1+\tau)z)}$$

DEF slash operator

$$F|_k\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(\tau) = (c\tau + d)^{-k} F\left(\frac{a\tau + b}{c\tau + d}\right)$$

Secant zeta function: Functional equation

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- In terms of

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

the functional equations become

$$\psi_{2m}|_{1-2m}(T^2-1) = 0,$$
$$\psi_{2m}|_{1-2m}(R^2-1) = \pi^{2m}\,\mathrm{rat}(\tau).$$

Secant zeta function: Functional equation

- The matrices

$$T^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad R^2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},$$

together with $-I$, generate

$$\Gamma(2) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \quad \gamma \equiv I \pmod{2}\}.$$

COR For any $\gamma \in \Gamma(2)$,

$$\psi_{2m}|_{1-2m}(\gamma - 1) = \pi^{2m} \operatorname{rat}(\tau).$$

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For positive integers m, r,

$$\psi_{2m}(\sqrt{r}) \in \mathbb{Q} \cdot \pi^{2m}.$$

proof

- Note that
$$\begin{pmatrix} X & rY \\ Y & X \end{pmatrix} \cdot \sqrt{r} = \sqrt{r}.$$

- As shown by Lagrange, there are X and Y which solve Pell's equation
$$X^2 - rY^2 = 1.$$

$$\psi_{2m}|_{1-2m}(\gamma - 1) = \pi^{2m}\,\mathrm{rat}(\tau).$$

$\square$

Secant zeta function: Special values

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For positive integers m, r,

$$\psi_{2m}(\sqrt{r}) \in \mathbb{Q} \cdot \pi^{2m}.$$

proof

- Note that
$$\begin{pmatrix} X & rY \\ Y & X \end{pmatrix} \cdot \sqrt{r} = \sqrt{r}.$$
- As shown by Lagrange, there are X and Y which solve Pell's equation
$$X^2 - rY^2 = 1.$$
- Since
$$\gamma = \begin{pmatrix} X & rY \\ Y & X \end{pmatrix}^2 = \begin{pmatrix} X^2 + rY^2 & 2rXY \\ 2XY & X^2 + rY^2 \end{pmatrix} \in \Gamma(2),$$
the claim follows from the evenness of ψ_{2m} and
$$\psi_{2m}|_{1-2m}(\gamma - 1) = \pi^{2m} \operatorname{rat}(\tau).$$

□

EG For integers κ, μ,

$$\psi_2\left(\kappa+\sqrt{\kappa\left(\tfrac{1}{\mu}+\kappa\right)}\right)=\frac{\pi^2}{6}\left(1+\frac{3\kappa}{2\mu}\right),$$

$$\psi_4\left(\kappa+\sqrt{\kappa\left(\tfrac{1}{\mu}+\kappa\right)}\right)=\frac{\pi^4}{90}\left(1+\frac{5\kappa}{2\mu}-\frac{5\kappa^2(16\mu^2-15)}{8\mu^2(4\kappa\mu+3)}\right).$$

PART II

Eichler integrals of Eisenstein series

$$\tilde{f}(\tau) = \int_{\tau}^{i\infty} [f(z) - a(0)] (z - \tau)^{k-2} \mathrm{d}z$$

Eichler integrals

$$D = \frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}\tau}$$

derivative

$$\partial_h = D - \frac{h}{4\pi y}$$

Maass raising operator
$\partial_h(F|_h\gamma) = (\partial_h F)|_{h+2}\gamma$

$$\partial_h^n = \partial_{h+2(n-1)} \circ \cdots \circ \partial_{h+2} \circ \partial_h$$

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$$\partial_h^n = \partial_{h+2(n-1)} \circ \cdots \circ \partial_{h+2} \circ \partial_h$$

- By induction on n, following Lewis–Zagier 2001,

$$\frac{\partial_h^n}{n!} = \sum_{j=0}^{n} \binom{n+h-1}{j} \left(-\frac{1}{4\pi y}\right)^j \frac{D^{n-j}}{(n-j)!}.$$

$$D = \frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}\tau}$$ derivative

$$\partial_h = D - \frac{h}{4\pi y}$$ Maass raising operator

$\partial_h(F|_h\gamma) = (\partial_h F)|_{h+2}\gamma$

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$$\frac{\partial_h^n}{n!} = \sum_{j=0}^n \binom{n+h-1}{j} \left(-\frac{1}{4\pi y}\right)^j \frac{D^{n-j}}{(n-j)!}.$$

- In the special case $n = 1-h$, with $h = 2-k$,

$$\partial_{2-k}^{k-1} = D^{k-1}.$$

Eichler integrals

THM Bol 1949

For all sufficiently differentiable F and all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$,

$$D^{k-1}(F|_{2-k}\gamma) = (D^{k-1}F)|_k\gamma.$$

EG $k = 2$

$$(DF)|_2\gamma = (c\tau + d)^{-2} F'\left(\frac{a\tau + b}{c\tau + d}\right) = D\left[F\left(\frac{a\tau + b}{c\tau + d}\right)\right]$$

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- F is an **Eichler integral** if $D^{k-1}F$ is modular of weight k.
- Then $D^{k-1}(F|_{2-k}\gamma) = D^{k-1}F$, and hence

$$F|_{2-k}(\gamma - 1) = \mathrm{poly}(\tau), \qquad \deg \mathrm{poly} \leqslant k-2.$$

- $\mathrm{poly}(\tau)$ is a **period polynomial** of the modular form.

Eichler integrals

- For modular $f(\tau) = \sum a(n) q^n$, weight k, define the **Eichler integral**

$$\tilde{f}(\tau) = \int_\tau^{i\infty} [f(z) - a(0)] (z - \tau)^{k-2} \mathrm{d}z$$
$$= \frac{(-1)^k \Gamma(k-1)}{(2\pi i)^{k-1}} \sum_{n=1}^{\infty} \frac{a(n)}{n^{k-1}} q^n.$$

If $a(0) = 0$, $\tilde{f}$ is an Eichler integral in the strict sense of the previous slide.

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If $a(0) = 0$, $\tilde{f}$ is an Eichler integral in the strict sense of the previous slide.

EG For cusp forms f of level 1, **the period polynomial** $\rho_f(X)$ is

$$\tilde{f}|_{2-k}(S-1) = \int_0^{i\infty} f(z)(z-X)^{k-2} \,\mathrm{d}z$$
$$= (-1)^k \sum_{s=1}^{k-1} \binom{k-2}{s-1} \frac{\Gamma(s)}{(2\pi i)^s} L(f,s) X^{k-s-1}.$$

The period polynomials

- Let $U = TS = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, and define

$$W_{k-2} = \left\{ \underset{\deg p = k-2}{p \in \mathbb{C}[X]} : p|_{2-k}(1+S) = p|_{2-k}(1+U+U^2) = 0 \right\}.$$

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- Denote with p^- the odd part of p.

THM Eichler–Shimura: The space of (level 1) cusp forms S_k is isomorphic to W_{k-2}^- via

$$f \mapsto \rho_f^-(X).$$

- Similarly, W_{k-2} is isomorphic to $S_k \oplus M_k$.

The period polynomials in higher level

- Let Γ be of finite index in $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$.
- Let V_{k-2} be the polynomials of degree at most $k-2$.

DEF
Pașol–Popa 2013

The **multiple period polynomial** of f is

$$\rho_f : \Gamma\backslash\Gamma_1 \to V_{k-2},$$

$$\rho_f(A)(X) = \int_0^{i\infty} [f|A(z) - a_0(f|A)]\,(z-X)^{k-2}\mathrm{d}z.$$

- $\gamma \in \Gamma_1$ acts on $p : \Gamma\backslash\Gamma_1 \to V_{k-2}$ by

$$p|\gamma(A) = p(A\gamma^{-1})|_{2-k}\gamma.$$

- Pașol and Popa extend Eichler–Shimura isomorphism to this settinsetting.

- For the **Eisenstein series** G_{2k},

$$G_{2k}(\tau) = 2\zeta(2k) + 2\frac{(2\pi i)^{2k}}{\Gamma(2k)} \underbrace{\sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n}_{\sum \frac{n^{2k-1}q^n}{1-q^n}},$$

$$\tilde{G}_{2k}(\tau) = \frac{4\pi i}{2k-1} \underbrace{\sum_{n=1}^{\infty} \frac{\sigma_{2k-1}(n)}{n^{2k-1}} q^n}_{\sum \frac{n^{1-2k}q^n}{1-q^n}}.$$

Eichler integrals of Eisenstein series

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- The period "polynomial" $\tilde{G}_{2k}|_{2-2k}(S-1)$ is given by

$$\frac{(2\pi i)^{2k}}{2k-1}\left[\sum_{s=0}^{k} \frac{B_{2s}}{(2s)!}\frac{B_{2k-2s}}{(2k-2s)!} X^{2s-1} + \frac{\zeta(2k-1)}{(2\pi i)^{2k-1}}(X^{2k-2}-1)\right].$$

Ramanujan's formula

THM Ramanujan, Grosswald

For $\alpha, \beta > 0$ such that $\alpha\beta = \pi^2$ and $m \in \mathbb{Z}$,

$$\alpha^{-m}\left\{\frac{\zeta(2m+1)}{2} + \sum_{n=1}^{\infty}\frac{n^{-2m-1}}{e^{2\alpha n}-1}\right\} = (-\beta)^{-m}\left\{\frac{\zeta(2m+1)}{2} + \sum_{n=1}^{\infty}\frac{n^{-2m-1}}{e^{2\beta n}-1}\right\}$$

$$-2^{2m}\sum_{n=0}^{m+1}(-1)^n\frac{B_{2n}}{(2n)!}\frac{B_{2m-2n+2}}{(2m-2n+2)!}\alpha^{m-n+1}\beta^n.$$

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$$-2^{2m}\sum_{n=0}^{m+1}(-1)^n\frac{B_{2n}}{(2n)!}\frac{B_{2m-2n+2}}{(2m-2n+2)!}\alpha^{m-n+1}\beta^n.$$

$$\frac{1}{e^x-1} = \frac{1}{2}\cot\left(\frac{x}{2}\right) - \frac{1}{2}$$

- In terms of

$$\xi_s(\tau) = \sum_{n=1}^{\infty}\frac{\cot(\pi n\tau)}{n^s},$$

Ramanujan's formula takes the form

$$\xi_{2k-1}|_{2-2k}(S-1) = (-1)^k(2\pi)^{2k-1}\sum_{s=0}^{k}\frac{B_{2s}}{(2s)!}\frac{B_{2k-2s}}{(2k-2s)!}\tau^{2s-1}.$$

Secant zeta function

- $\sum \frac{\cot(\pi n \tau)}{n^{2k-1}}$ is an Eichler integral of the Eisenstein series G_{2k}.

EG

$$\cot(\pi\tau) = \frac{1}{\pi} \sum_{j \in \mathbb{Z}} \frac{1}{\tau + j}$$

$$\lim_{N \to \infty} \sum_{j=-N}^{N}$$

Secant zeta function

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$$\cot(\pi\tau) = \frac{1}{\pi} \sum_{j\in\mathbb{Z}} \frac{1}{\tau + j}$$

$$\lim_{N\to\infty} \sum_{j=-N}^{N}$$

- $\sum \frac{\sec(\pi n\tau)}{n^{2k}}$ is an Eichler integral of an Eisenstein series with character.

EG

$$\sec\left(\frac{\pi\tau}{2}\right) = \frac{2}{\pi} \sum_{j\in\mathbb{Z}} \frac{\chi_{-4}(j)}{\tau + j}$$

- $\sum_{m,n\in\mathbb{Z}}' \frac{\chi_{-4}(n)}{(m\tau + n)^{2k+1}}$ is an Eisenstein series of weight $2k + 1$.

- More generally, we have the Eisenstein series

$$E_k(\tau;\chi,\psi) = \sum_{m,n\in\mathbb{Z}}{}' \frac{\chi(m)\psi(n)}{(m\tau+n)^k},$$

where χ and ψ are Dirichlet characters modulo L and M.

- We assume $\chi(-1)\psi(-1) = (-1)^k$. Otherwise, $E_k(\tau;\chi,\psi) = 0$.

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- We assume $\chi(-1)\psi(-1) = (-1)^k$. Otherwise, $E_k(\tau;\chi,\psi) = 0$.

PROP Modular transformations: $\gamma = \left(\begin{smallmatrix} a & Mb \\ Lc & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$

- $E_k(\tau;\chi,\psi)|_k\gamma = \chi(d)\bar{\psi}(d)E_k(\tau;\chi,\psi)$
- $E_k(\tau;\chi,\psi)|_k S = \chi(-1)E_k(\tau;\psi,\chi)$

PROP If ψ is primitive, the L-function of $E(\tau) = E_k(\tau;\chi,\psi)$ is

$$L(E,s) = \mathrm{const}\cdot M^s L(\chi,s)L(\bar{\psi},1-k+s).$$

EG
Euler

$$\zeta(2n) = -\frac{1}{2}(2\pi i)^{2n}\frac{B_{2n}}{(2n)!}$$

- For integer $n > 0$ and primitive χ with $\chi(-1) = (-1)^n$,
  (χ of conductor L and Gauss sum $G(\chi)$)

$$L(n, \chi) = (-1)^{n-1}\frac{G(\chi)}{2}\left(\frac{2\pi i}{L}\right)^n \frac{B_{n,\bar{\chi}}}{n!},$$
$$L(1-n, \chi) = -B_{n,\chi}/n.$$

- The **generalized Bernoulli numbers** have generating function

$$\sum_{n=0}^{\infty} B_{n,\chi}\frac{x^n}{n!} = \sum_{a=1}^{L}\frac{\chi(a)xe^{ax}}{e^{Lx}-1}.$$

Period polynomials of Eisenstein series

THM Berndt-S 2013

For $k \geqslant 3$ and primitive $\chi \neq 1$, $\psi \neq 1$,

$$\tilde{E}_k(X;\chi,\psi) - \psi(-1)X^{k-2}\tilde{E}_k(-1/X;\psi,\chi) = \text{const} \sum_{s=0}^{k} \frac{B_{k-s,\bar{\chi}}}{(k-s)!L^{k-s}} \frac{B_{s,\bar{\psi}}}{s!M^s} X^{s-1}.$$

$$\text{const} = -\chi(-1)G(\chi)\,G(\psi)\frac{(2\pi i)^k}{k-1}$$

THM Berndt-S 2013

For $k \geqslant 3$ and primitive $\chi \neq 1$, $\psi \neq 1$,

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$$\text{const} = -\chi(-1)G(\chi)\,G(\psi)\frac{(2\pi i)^k}{k-1}$$

- If χ or ψ are principal, then we need to add to the RHS:

$$-\frac{2\psi(-1)}{k-1}\pi i \left[\delta_{\chi=1} L(k-1,\psi)X^{k-2} - \delta_{\psi=1} L(k-1,\chi)\right]$$

- Recall that we assume $\chi(-1)\psi(-1) = (-1)^k$.

COR Berndt-S 2013

For $k \geqslant 3$, primitive χ, $\psi \neq 1$, and n such that $L|n$,

$R^n = \left(\begin{smallmatrix} 1 & 0 \\ n & 1 \end{smallmatrix}\right)$

$$\tilde{E}_k(X;\chi,\psi)|_{2-k}(1-R^n)$$
$$= \text{const} \sum_{s=0}^{k} \frac{B_{k-s,\bar{\chi}}}{(k-s)!L^{k-s}} \frac{B_{s,\bar{\psi}}}{s!M^s} X^{s-1}|_{2-k}(1-R^n).$$

- Note that

$$X^{s-1}|_{2-k}(1-R^n) = X^{s-1}(1-(nX+1)^{k-1-s}).$$

Application: Grosswald-type formula for Dirichlet L-values

THM Berndt-S 2013

For $\alpha \in \mathcal{H}$, such that $R_k(\alpha; \bar{\chi}, 1) = 0$ and $\alpha^{k-2} \neq 1$,
($k \geqslant 3$, χ primitive, $\chi(-1) = (-1)^k$)

$$\begin{aligned} L(k-1, \chi) &= \frac{k-1}{2\pi i(1-\alpha^{k-2})} \left[\tilde{E}_k \left(\frac{\alpha - 1}{L}; \chi, 1 \right) - \alpha^{k-2} \tilde{E}_k \left(\frac{1 - 1/\alpha}{L}; \chi, 1 \right) \right] \\ &= \frac{2}{1-\alpha^{k-2}} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{k-1}} \left[\frac{1}{1 - e^{2\pi i n(1-\alpha)/L}} - \frac{\alpha^{k-2}}{1 - e^{2\pi i n(1/\alpha - 1)/L}} \right]. \end{aligned}$$

Application: Grosswald-type formula for Dirichlet L-values

THM Berndt-S 2013

For $\alpha \in \mathcal{H}$, such that $R_k(\alpha; \bar{\chi}, 1) = 0$ and $\alpha^{k-2} \neq 1$, ($k \geqslant 3$, χ primitive, $\chi(-1) = (-1)^k$)

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$$= \frac{2}{1 - \alpha^{k-2}} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{k-1}} \left[\frac{1}{1 - e^{2\pi i n(1-\alpha)/L}} - \frac{\alpha^{k-2}}{1 - e^{2\pi i n(1/\alpha - 1)/L}} \right].$$

THM Gun–Murty–Rath 2011

As $\beta \in \mathcal{H}$, $\beta^{2k-2} \neq 1$, ranges over algebraic numbers, the values

$$\frac{1}{\pi} \left[\tilde{E}_{2k}(\beta; 1, 1) - \beta^{2k-2} \tilde{E}_{2k}(-1/\beta; 1, 1) \right]$$

contain at most one algebraic number.

PART III

Unimodularity of period polynomials

$$R_k(X) = \sum_{s=0}^{k} \frac{B_s}{s!} \frac{B_{k-s}}{(k-s)!} X^{s-1}$$

Unimodular polynomials

DEF $p(x)$ is **unimodular** if all its zeros have absolute value 1.

- Kronecker: if $p(x) \in \mathbb{Z}[x]$ is monic and unimodular, hence Mahler measure 1, then all of its roots are roots of unity.

EG Lehmer

$$x^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$$

has only the two real roots 0.850, 1.176 off the unit circle.

Lehmer's conjecture: $1.176\ldots$ is the smallest Mahler measure (greater than 1)

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$$x^2 + \frac{6}{5}x + 1 = \left(x + \frac{3+4i}{5}\right)\left(x + \frac{3-4i}{5}\right)$$

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THM Cohn 1922

$P(x)$ is unimodular if and only if

- $P(x) = a_0 + a_1 x + \ldots + a_n x^n$ is self-inversive, i.e. $a_k = \varepsilon \overline{a_{n-k}}$ for some $|\varepsilon| = 1$, and
- $P'(x)$ has all its roots within the unit circle.

- Following Gun–Murty–Rath, the **Ramanujan polynomials** are

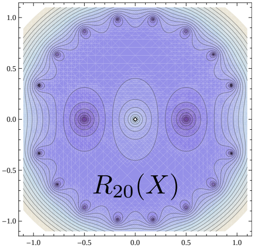
$$R_k(X) = \sum_{s=0}^{k} \frac{B_s}{s!} \frac{B_{k-s}}{(k-s)!} X^{s-1}.$$

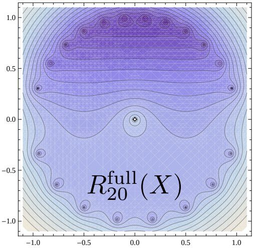
THM Murty-Smyth-Wang '11

All nonreal zeros of $R_k(X)$ lie on the unit circle.
For $k \geqslant 2$, $R_{2k}(X)$ has exactly four real roots which approach $\pm 2^{\pm 1}$.

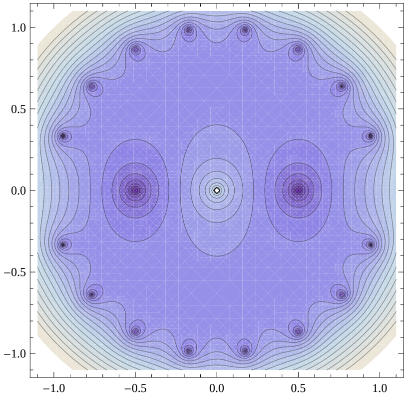
THM Lalín-Smyth '13

$R_{2k}(X) + \dfrac{\zeta(2k-1)}{(2\pi i)^{2k-1}}(X^{2k-2} - 1)$ is unimodular.

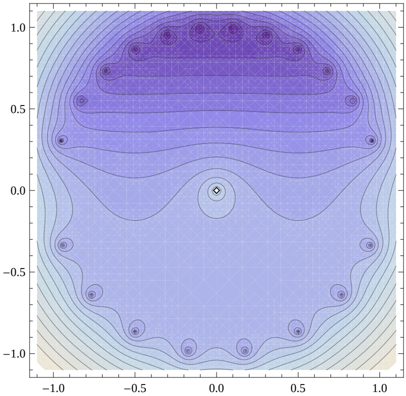




Ramanujan polynomials

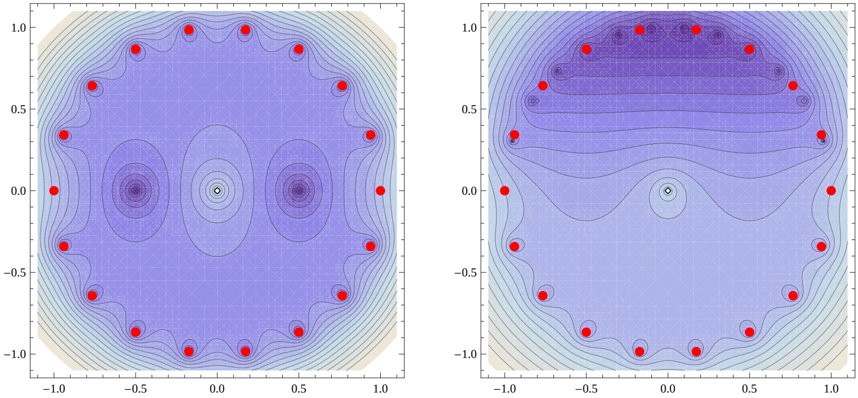


$R_{20}(X)$



$R_{20}^{\mathrm{full}}(X)$

Ramanujan polynomials



$R_{20}(X)$ $\qquad$ $R_{20}^{\text{full}}(X)$

Generalized Ramanujan polynomials

- Consider the following **generalized Ramanujan polynomials**:

$$R_k(X) = \sum_{s=0}^{k} \frac{B_s}{s!} \frac{B_{k-s}}{(k-s)!} X^{s-1}$$

$$R_k(X;\chi,\psi) = \sum_{s=0}^{k} \frac{B_{s,\chi}}{s!} \frac{B_{k-s,\psi}}{(k-s)!} \left(\frac{X-1}{M}\right)^{k-s-1} (1 - X^{s-1})$$

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PROP Berndt-S 2013

- For $k > 1$, $R_{2k}(X;1,1) = R_{2k}(X)$.
- $R_k(X;\chi,\psi)$ is self-inversive.

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PROP Berndt-S 2013

- For $k > 1$, $R_{2k}(X;1,1) = R_{2k}(X)$.
- $R_k(X;\chi,\psi)$ is self-inversive.

CONJ If χ, ψ are nonprincipal real, then $R_k(X;\chi,\psi)$ is unimodular.

Generalized Ramanujan polynomials

EG

$$R_k(X;\chi,1)$$

For χ real, apparently unimodular unless:

- $\chi = 1$: $R_{2k}(X;1,1)$ has real roots approaching $\pm 2^{\pm 1}$
- $\chi = 3-$: $R_{2k+1}(X;3-,1)$ has real roots approaching $-2^{\pm 1}$

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EG

$$R_k(X;1,\psi)$$

Apparently:

- unimodular for ψ one of
$$3-, 4-, 5+, 8\pm, 11-, 12+, 13+, 19-, 21+, 24+, \ldots$$
- all nonreal roots on the unit circle if ψ is one of
$$1+, 7-, 15-, 17+, 20-, 23-, 24-, \ldots$$
- four nonreal zeros off the unit circle if ψ is one of
$$35-, 59-, 83-, 131-, 155-, 179-, \ldots$$

- A second kind of **generalized Ramanujan polynomials**:

$$R_k(X) = \sum_{s=0}^{k} \frac{B_s}{s!} \frac{B_{k-s}}{(k-s)!} X^{s-1}$$

$$S_k(X;\chi,\psi) = \sum_{s=0}^{k} \frac{B_{s,\chi}}{s!} \frac{B_{k-s,\psi}}{(k-s)!} \left(\frac{LX}{M}\right)^{k-s-1}$$

- Obviously, $S_k(X;1,1) = R_k(X)$.

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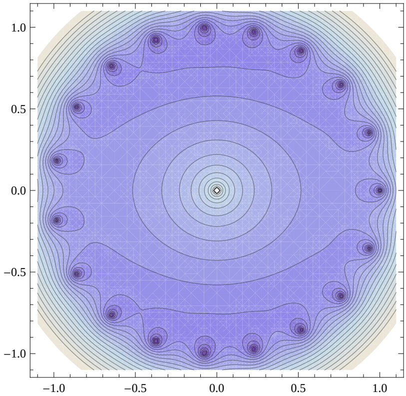
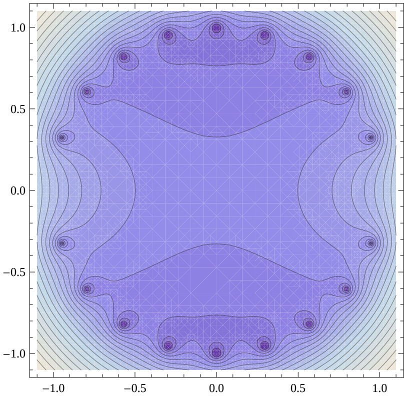
- Obviously, $S_k(X;1,1) = R_k(X)$.

CONJ If χ is nonprincipal real, then $S_k(X;\chi,\chi)$ is unimodular (up to trivial zero roots).

Generalized Ramanujan polynomials

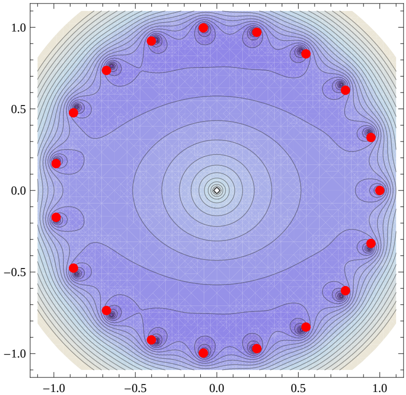


$R_{19}(X; 1, \chi_{-4})$



$S_{20}(X; \chi_{-4}, \chi_{-4})$

Generalized Ramanujan polynomials



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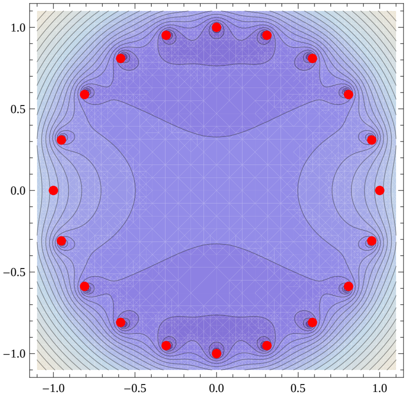

$S_{20}(X; \chi_{-4}, \chi_{-4})$

Unimodularity of period polynomials

- Both kinds of generalized Ramanujan polynomials are, essentially, period polynomials: χ, ψ primitive, nonprincipal

$$\begin{aligned} S_k(X;\chi,\psi) &= \mathrm{const}\cdot\left[\tilde{E}_k(X;\bar{\chi},\bar{\psi}) - \psi(-1)X^{k-2}\tilde{E}_k(-1/X;\bar{\psi},\bar{\chi})\right] \\ R_k(LX+1;\chi,\psi) &= S_k(X;\chi,\psi)|_{2-k}(1-R^L) \\ &= \mathrm{const}\cdot\tilde{E}_k(X;\bar{\chi},\bar{\psi})|_{2-k}(1-R^L) \end{aligned}$$

Unimodularity of period polynomials

- Both kinds of generalized Ramanujan polynomials are, essentially, period polynomials: χ, ψ primitive, nonprincipal

$$S_k(X;\chi,\psi) = \mathrm{const} \cdot \left[\tilde{E}_k(X;\bar{\chi},\bar{\psi}) - \psi(-1)X^{k-2}\tilde{E}_k(-1/X;\bar{\psi},\bar{\chi})\right]$$

$$R_k(LX+1;\chi,\psi) = S_k(X;\chi,\psi)|_{2-k}(1-R^L) = \mathrm{const} \cdot \tilde{E}_k(X;\bar{\chi},\bar{\psi})|_{2-k}(1-R^L)$$

THM Conrey-Farmer-Imamoglu 2012: For any Hecke cusp form (for $\mathrm{SL}_2(\mathbb{Z})$), the odd part of its period polynomial has

- trivial zeros at 0, ± 2, $\pm\frac{1}{2}$,
- and all remaining zeros lie on the unit circle.

THM El-Guindy–Raji 2013: For any Hecke eigenform (for $\mathrm{SL}_2(\mathbb{Z})$), the full period polynomial has all zeros on the unit circle.

THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`

B. Berndt, A. Straub
On a secant Dirichlet series and Eichler integrals of Eisenstein series
Preprint, 2013