Arithmetic aspects of short random walks

Number Theory Seminar

Armin Straub

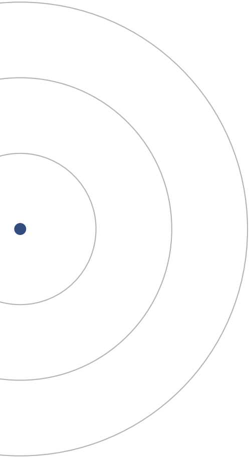
September 27, 2012

University of Illinois
at Urbana–Champaign

Based on joint work with:

Jon Borwein James Wan Wadim Zudilin
University of Newcastle, Australia

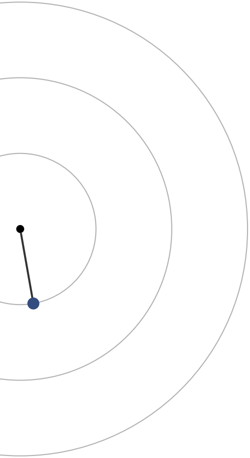
Random walks



- n-step uniform planar random walk in the plane:
  - n steps, each of length 1,
  - taken in randomly chosen direction

Q What is the distance traveled in n steps?

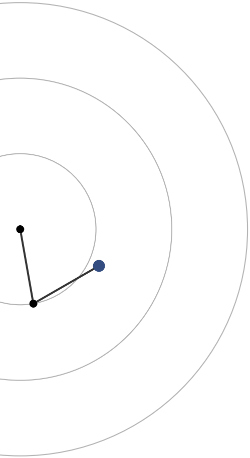
$p_n(x)$	probability density
$W_n(s)$	sth moment



- n-step uniform planar random walk in the plane:
  - n steps, each of length 1,
  - taken in randomly chosen direction

Q What is the distance traveled in n steps?

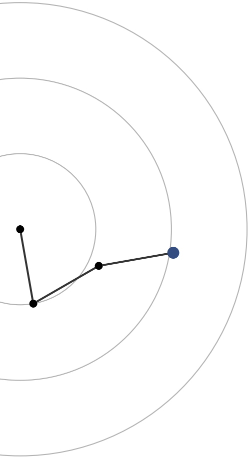
$p_n(x)$	probability density
$W_n(s)$	sth moment



- n-step uniform planar random walk in the plane:
  - n steps, each of length 1,
  - taken in randomly chosen direction

Q What is the distance traveled in n steps?

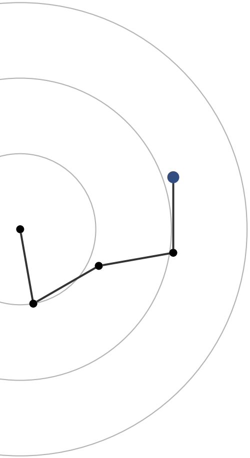
$p_n(x)$	probability density
$W_n(s)$	sth moment



- n-step uniform planar random walk in the plane:
  - n steps, each of length 1,
  - taken in randomly chosen direction

Q What is the distance traveled in n steps?

$p_n(x)$	probability density
$W_n(s)$	sth moment

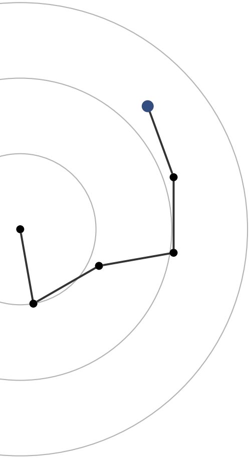


- n-step uniform planar random walk in the plane:
  - n steps, each of length 1,
  - taken in randomly chosen direction

Q What is the distance traveled in n steps?

$p_n(x)$	probability density
$W_n(s)$	sth moment

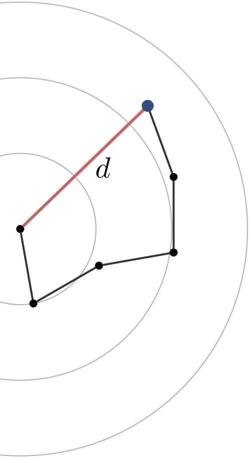
Random walks



- n-step uniform planar random walk in the plane:
  - n steps, each of length 1,
  - taken in randomly chosen direction

Q	What is the distance traveled in n steps?
	$p_n(x)$ probability density
	$W_n(s)$ sth moment

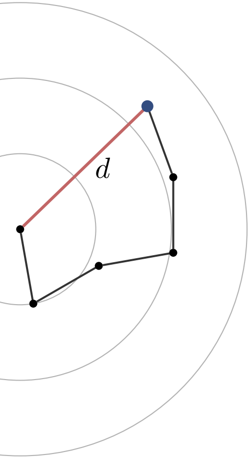
Random walks



- n-step uniform planar random walk in the plane:
  - n steps, each of length 1,
  - taken in randomly chosen direction

Q What is the distance traveled in n steps?

$p_n(x)$	probability density
$W_n(s)$	sth moment



- n-step uniform planar random walk in the plane:
  - n steps, each of length 1,
  - taken in randomly chosen direction

Q What is the distance traveled in n steps?

$p_n(x)$ probability density

$W_n(s)$ sth moment

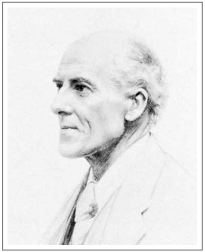
EG

$$W_2(1) = \frac{4}{\pi}$$

Random walks are only about 100 years old

- Karl Pearson asked for $p_n(x)$ in Nature in 1905.

  This famous question coined the term **random walk**.



The Problem of the Random Walk.

Can any of your readers refer me to a work wherein I should find a solution of the following problem, or failing the knowledge of any existing solution provide me with an original one? I should be extremely grateful for aid in the matter.

A man starts from a point O and walks l yards in a straight line; he then turns through any angle whatever and walks another l yards in a second straight line. He repeats this process n times. I require the probability that after these n stretches he is at a distance between r and $r+\delta r$ from his starting point, O.

The problem is one of considerable interest, but I have only succeeded in obtaining an integrated solution for *two* stretches. I think, however, that a solution ought to be found, if only in the form of a series in powers of $1/n$, when n is large.

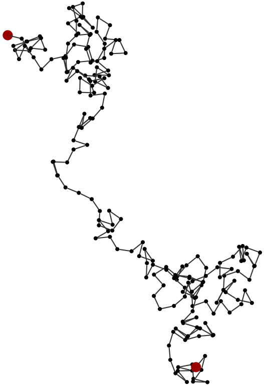
Karl Pearson.

The Gables, East Ilsley, Berks.

Applications include:

- dispersion of mosquitoes
- random migration of micro-organisms
- phenomenon of laser speckle

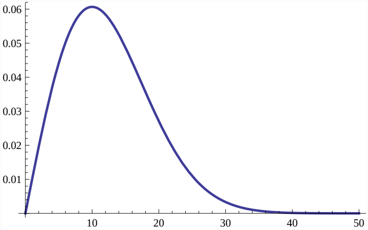
Long random walks



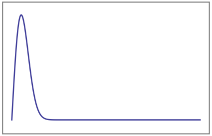
THM Rayleigh, 1905

$$p_n(x) \approx \frac{2x}{n} e^{-x^2/n} \qquad \text{for large } n$$

EG p_{200}

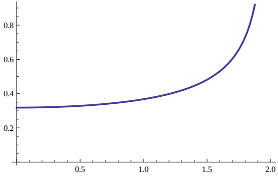


> *The lesson of Lord Rayleigh's solution is that in open country the most probable place to find a drunken man who is at all capable of keeping on his feet is somewhere near his starting point!*
>
> **Karl Pearson**, *1905*

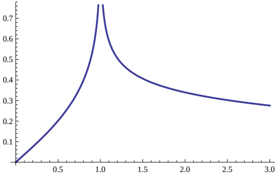


Densities of short walks

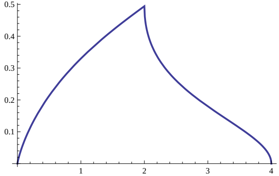
p_2



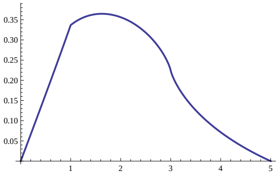
p_3



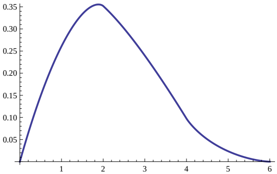
p_4



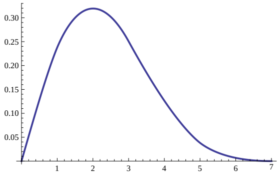
p_5



p_6

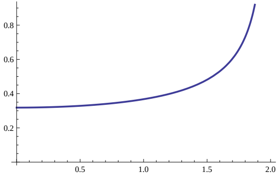


p_7

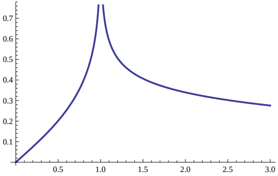


Densities of short walks

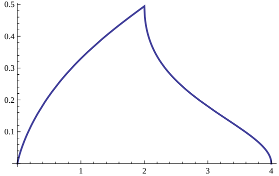
p_2



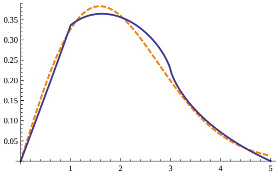
p_3



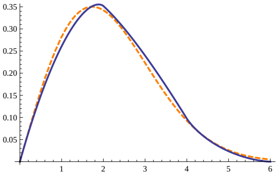
p_4



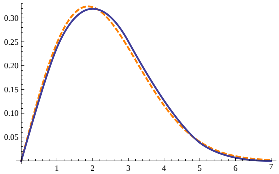
p_5



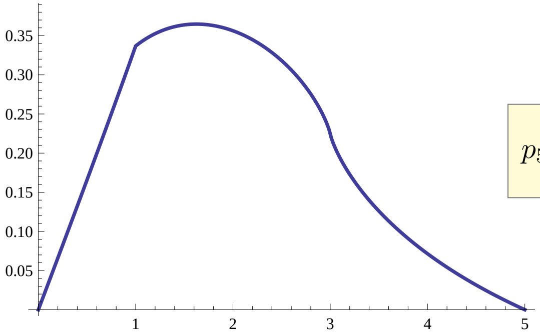
p_6



p_7



The density of a five-step random walk



$$p_5(x) = \int_0^\infty xtJ_0(xt)J_0^5(t)\,\mathrm{d}t$$

> *...the graphical construction, however carefully reinvestigated, did not permit of our considering the curve to be anything but a* **straight** *line... Even if it is not absolutely true, it exemplifies the extraordinary power of such integrals of J products to give extremely close approximations to such simple forms as horizontal lines.*
>
> **Karl Pearson**, *1906*

H. E. Fettis
On a conjecture of Karl Pearson
Rider Anniversary Volume, p. 39–54, 1963

Classical results on the densities

$$p_2(x) = \frac{2}{\pi\sqrt{4-x^2}}$$ easy

$$p_3(x) = \operatorname{Re}\left(\frac{\sqrt{x}}{\pi^2}\,K\left(\sqrt{\frac{(x+1)^3(3-x)}{16x}}\right)\right)$$ G. J. Bennett 1905

$$p_4(x) = \;??$$

$$\vdots$$

$$p_n(x) = \int_0^\infty xtJ_0(xt)J_0^n(t)\,\mathrm{d}t$$ J. C. Kluyver 1906

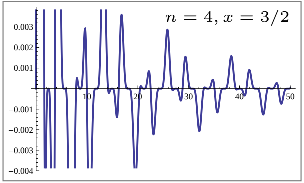
$$p_2(x) = \frac{2}{\pi\sqrt{4-x^2}}$$ easy

$$p_3(x) = \mathrm{Re}\left(\frac{\sqrt{x}}{\pi^2} K\left(\sqrt{\frac{(x+1)^3(3-x)}{16x}}\right)\right)$$ G. J. Bennett 1905

$$p_4(x) = ??$$

$$\vdots$$

$$p_n(x) = \int_0^\infty xt J_0(xt) J_0^n(t)\,\mathrm{d}t$$ J. C. Kluyver 1906



An exact probability

THM The probability that a random walk is within one unit from its origin after n steps is ...?

An exact probability

THM The probability that a random walk is within one unit from its origin after n steps is $\frac{1}{n+1}$.

An exact probability

THM The probability that a random walk is within one unit from its origin after n steps is $\frac{1}{n+1}$.

Proof. The cumulative density function P_n can be expressed as

$$P_n(x) = \int_0^\infty x J_1(xt) J_0^n(t) \,\mathrm{d}t.$$

Then:

$$P_n(1) = \frac{J_0(0)^{n+1}}{n+1} = \frac{1}{n+1}.$$

□

- Recently: remarkably short proof by Olivier Bernardi

The average distance traveled in two steps

- The average distance in two steps:

$$W_2(1) = \int_0^1 \int_0^1 \left| e^{2\pi i x} + e^{2\pi i y} \right| \mathrm{d}x \mathrm{d}y = \ ?$$

The average distance traveled in two steps

- The average distance in two steps:

$$W_2(1) = \int_0^1 \int_0^1 \left| e^{2\pi i x} + e^{2\pi i y} \right| \mathrm{d}x\mathrm{d}y = \; ?$$
$$= \int_0^1 \left| 1 + e^{2\pi i y} \right| \mathrm{d}y$$

The average distance traveled in two steps

- The average distance in two steps:

$$W_2(1) = \int_0^1 \int_0^1 \left| e^{2\pi i x} + e^{2\pi i y} \right| \mathrm{d}x\mathrm{d}y = \ ?$$

$$= \int_0^1 \left| 1 + e^{2\pi i y} \right| \mathrm{d}y$$

$$= \int_0^1 2\cos(\pi y)\mathrm{d}y$$

$$\begin{aligned} &\left|1 + e^{2\pi i y}\right| \\ &= |1 + \cos(2\pi y) + i\sin(2\pi y)| \\ &= \sqrt{2 + 2\cos(2\pi y)} \\ &= 2\cos(\pi y) \end{aligned}$$

The average distance traveled in two steps

- The average distance in two steps:

$$W_2(1) = \int_0^1 \int_0^1 \left|e^{2\pi i x} + e^{2\pi i y}\right| \mathrm{d}x\mathrm{d}y = \ ?$$

$$= \int_0^1 \left|1 + e^{2\pi i y}\right| \mathrm{d}y$$

$$= \int_0^1 2\cos(\pi y)\mathrm{d}y$$

$$= \frac{4}{\pi} \approx 1.27324$$

$$\left|1 + e^{2\pi i y}\right|$$
$$= |1 + \cos(2\pi y) + i\sin(2\pi y)|$$
$$= \sqrt{2 + 2\cos(2\pi y)}$$
$$= 2\cos(\pi y)$$

The average distance traveled in two steps

- The average distance in two steps:

$$W_2(1) = \int_0^1 \int_0^1 \left| e^{2\pi i x} + e^{2\pi i y} \right| \mathrm{d}x \mathrm{d}y = \ ?$$

$$= \int_0^1 \left| 1 + e^{2\pi i y} \right| \mathrm{d}y$$

$$= \int_0^1 2\cos(\pi y) \mathrm{d}y$$

$$= \frac{4}{\pi} \approx 1.27324$$

$$\begin{aligned} &\left|1 + e^{2\pi i y}\right| \\ &= |1 + \cos(2\pi y) + i\sin(2\pi y)| \\ &= \sqrt{2 + 2\cos(2\pi y)} \\ &= 2\cos(\pi y) \end{aligned}$$

- Mathematica 7 and Maple 14 think the double integral is 0.

  Better: Mathematica 8 just does not evaluate the integral.

The average distance traveled in two steps

- The average distance in two steps:

$$W_2(1) = \int_0^1 \int_0^1 \left| e^{2\pi i x} + e^{2\pi i y} \right| \mathrm{d}x\mathrm{d}y = \; ?$$

$$= \int_0^1 \left| 1 + e^{2\pi i y} \right| \mathrm{d}y$$

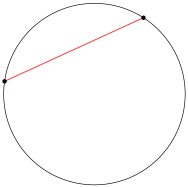
$$= \int_0^1 2\cos(\pi y)\mathrm{d}y$$

$$= \frac{4}{\pi} \approx 1.27324$$

$$\begin{aligned} &\left|1 + e^{2\pi i y}\right| \\ &= \left|1 + \cos(2\pi y) + i\sin(2\pi y)\right| \\ &= \sqrt{2 + 2\cos(2\pi y)} \\ &= 2\cos(\pi y) \end{aligned}$$

- Mathematica 7 and Maple 14 think the double integral is 0.

  Better: Mathematica 8 just does not evaluate the integral.
- This is the average length of a random arc on a unit circle.



Moments of random walks

DEF The sth moment $W_n(s)$ of the density p_n:

$$W_n(s) := \int_0^\infty x^s p_n(x)\,\mathrm{d}x = \int_{[0,1]^n} \left|e^{2\pi i x_1} + \ldots + e^{2\pi i x_n}\right|^s \mathrm{d}\boldsymbol{x}$$

Moments of random walks

DEF The sth moment $W_n(s)$ of the density p_n:

$$W_n(s) := \int_0^\infty x^s p_n(x)\,\mathrm{d}x = \int_{[0,1]^n} \left|e^{2\pi i x_1} + \ldots + e^{2\pi i x_n}\right|^s \mathrm{d}\boldsymbol{x}$$

- On a desktop:

$$\begin{aligned} W_3(1) &\approx 1.57459723755189365749 \\ W_4(1) &\approx 1.79909248 \\ W_5(1) &\approx 2.00816 \end{aligned}$$

Moments of random walks

DEF The sth moment $W_n(s)$ of the density p_n:

$$W_n(s) := \int_0^\infty x^s p_n(x)\,\mathrm{d}x = \int_{[0,1]^n} \left| e^{2\pi i x_1} + \ldots + e^{2\pi i x_n} \right|^s \mathrm{d}\boldsymbol{x}$$

- On a desktop:

$$\begin{aligned} W_3(1) &\approx 1.57459723755189365749 \\ W_4(1) &\approx 1.79909248 \\ W_5(1) &\approx 2.00816 \end{aligned}$$

- On a supercomputer: Lawrence Berkeley National Laboratory, 256 cores

$$W_5(1) \approx 2.0081618$$

Moments of random walks

DEF The sth moment $W_n(s)$ of the density p_n:

$$W_n(s) := \int_0^\infty x^s p_n(x)\,\mathrm{d}x = \int_{[0,1]^n} \left|e^{2\pi i x_1} + \ldots + e^{2\pi i x_n}\right|^s \mathrm{d}\boldsymbol{x}$$

- On a desktop:

$$\begin{aligned} W_3(1) &\approx 1.57459723755189365749 \\ W_4(1) &\approx 1.79909248 \\ W_5(1) &\approx 2.00816 \end{aligned}$$

- On a supercomputer: Lawrence Berkeley National Laboratory, 256 cores

$$W_5(1) \approx 2.0081618$$

- Hard to evaluate numerically to high precision.
  Monte-Carlo integration gives approximations with an asymptotic error of $O(1/\sqrt{N})$ where N is the number of sample points.

DEF The sth moment $W_n(s)$ of the density p_n:

$$W_n(s) := \int_0^\infty x^s p_n(x)\,\mathrm{d}x = \int_{[0,1]^n} \left|e^{2\pi i x_1} + \ldots + e^{2\pi i x_n}\right|^s \mathrm{d}\boldsymbol{x}$$

n	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$	$s=7$
2	1.273	2.000	3.395	6.000	10.87	20.00	37.25
3	1.575	3.000	6.452	15.00	36.71	93.00	241.5
4	1.799	4.000	10.12	28.00	82.65	256.0	822.3
5	2.008	5.000	14.29	45.00	152.3	545.0	2037.
6	2.194	6.000	18.91	66.00	248.8	996.0	4186.

Moments of random walks

DEF The sth moment $W_n(s)$ of the density p_n:

$$W_n(s) := \int_0^\infty x^s p_n(x)\,\mathrm{d}x = \int_{[0,1]^n} \left|e^{2\pi i x_1} + \ldots + e^{2\pi i x_n}\right|^s \mathrm{d}\boldsymbol{x}$$

n	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$	$s=7$
2	1.273	2.000	3.395	6.000	10.87	20.00	37.25
3	1.575	3.000	6.452	15.00	36.71	93.00	241.5
4	1.799	4.000	10.12	28.00	82.65	256.0	822.3
5	2.008	5.000	14.29	45.00	152.3	545.0	2037.
6	2.194	6.000	18.91	66.00	248.8	996.0	4186.

$W_2(1) = \frac{4}{\pi}$

Moments of random walks

DEF The sth moment $W_n(s)$ of the density p_n:

$$W_n(s) := \int_0^\infty x^s p_n(x)\,\mathrm{d}x = \int_{[0,1]^n} \left| e^{2\pi i x_1} + \ldots + e^{2\pi i x_n} \right|^s \mathrm{d}\boldsymbol{x}$$

n	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$	$s=7$
2	1.273	2.000	3.395	6.000	10.87	20.00	37.25
3	1.575	3.000	6.452	15.00	36.71	93.00	241.5
4	1.799	4.000	10.12	28.00	82.65	256.0	822.3
5	2.008	5.000	14.29	45.00	152.3	545.0	2037.
6	2.194	6.000	18.91	66.00	248.8	996.0	4186.

$W_2(1) = \frac{4}{\pi}$

$W_3(1) = 1.57459723755189\ldots = ?$

Moments of random walks

DEF The sth moment $W_n(s)$ of the density p_n:

$$W_n(s) := \int_0^\infty x^s p_n(x)\,\mathrm{d}x = \int_{[0,1]^n} \left|e^{2\pi i x_1} + \ldots + e^{2\pi i x_n}\right|^s \mathrm{d}\boldsymbol{x}$$

n	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$	$s=7$
2	1.273	2.000	3.395	6.000	10.87	20.00	37.25
3	1.575	3.000	6.452	15.00	36.71	93.00	241.5
4	1.799	4.000	10.12	28.00	82.65	256.0	822.3
5	2.008	5.000	14.29	45.00	152.3	545.0	2037.
6	2.194	6.000	18.91	66.00	248.8	996.0	4186.

$W_2(1) = \frac{4}{\pi}$

$W_3(1) = 1.57459723755189\ldots = ?$

Even moments

n	$s=0$	$s=2$	$s=4$	$s=6$	$s=8$	$s=10$	Sloane's
2	1	2	6	20	70	252	A000984
3	1	3	15	93	639	4653	A002893
4	1	4	28	256	2716	31504	A002895
5	1	5	45	545	7885	127905	A169714
6	1	6	66	996	18306	384156	A169715

Even moments

n	$s=0$	$s=2$	$s=4$	$s=6$	$s=8$	$s=10$	Sloane's
2	1	2	6	20	70	252	A000984
3	1	3	15	93	639	4653	A002893
4	1	4	28	256	2716	31504	A002895
5	1	5	45	545	7885	127905	A169714
6	1	6	66	996	18306	384156	A169715

EG

$$W_3(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j}$$

$$W_4(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} \binom{2(k-j)}{k-j}$$

$$W_5(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2(k-j)}{k-j} \sum_{\ell=0}^{j} \binom{j}{\ell}^2 \binom{2\ell}{\ell}$$

A combinatorial formula for the even moments

- sth moment $W_n(s)$ of the density p_n:

$$W_n(s) = \int_{[0,1]^n} \left| e^{2\pi i x_1} + \ldots + e^{2\pi i x_n} \right|^s \mathrm{d}\boldsymbol{x}$$

THM Borwein-Nuyens-S-Wan 2010

$$W_n(2k) = \sum_{a_1+\cdots+a_n=k} \binom{k}{a_1, \ldots, a_n}^2$$

A combinatorial formula for the even moments

- sth moment $W_n(s)$ of the density p_n:

$$W_n(s) = \int_{[0,1]^n} \left| e^{2\pi i x_1} + \ldots + e^{2\pi i x_n} \right|^s \mathrm{d}\boldsymbol{x}$$

THM Borwein-Nuyens-S-Wan 2010

$$W_n(2k) = \sum_{a_1+\cdots+a_n=k} \binom{k}{a_1, \ldots, a_n}^2$$

- $W_n(2k)$ counts the number of **abelian squares**: strings xy of length $2k$ from an alphabet with n letters such that y is a permutation of x.

A combinatorial formula for the even moments

- sth moment $W_n(s)$ of the density p_n:

$$W_n(s) = \int_{[0,1]^n} \left| e^{2\pi i x_1} + \ldots + e^{2\pi i x_n} \right|^s \mathrm{d}\boldsymbol{x}$$

THM Borwein-Nuyens-S-Wan 2010

$$W_n(2k) = \sum_{a_1+\cdots+a_n=k} \binom{k}{a_1, \ldots, a_n}^2$$

- $W_n(2k)$ counts the number of **abelian squares**: strings xy of length $2k$ from an alphabet with n letters such that y is a permutation of x.
- Introduced by Erdős and studied by others.

EG $acbc\, ccba$ is an abelian square. It contributes to $f_3(4)$.

L. B. Richmond and J. Shallit
Counting abelian squares
The Electronic Journal of Combinatorics, Vol. 16, 2009.

Moments of a two-step walk

EG $W_2(2k)$: abelian squares of length $2k$ from 2 letters

$$b\ a\ b\ a\ a \quad a\ b\ a\ a\ b$$

Moments of a two-step walk

EG $W_2(2k)$: abelian squares of length $2k$ from 2 letters

$$b\ \boldsymbol{a}\ b\ \boldsymbol{a}\ \boldsymbol{a} \quad a\ \boldsymbol{b}\ a\ a\ \boldsymbol{b}$$

Moments of a two-step walk

EG $W_2(2k)$: abelian squares of length $2k$ from 2 letters

$$b\ \boldsymbol{a}\ b\ \boldsymbol{a}\ \boldsymbol{a} \qquad a\ \boldsymbol{b}\ a\ a\ \boldsymbol{b}$$

Hence $W_2(2k) = \binom{2k}{k}$.

Moments of a two-step walk

EG $W_2(2k)$: abelian squares of length $2k$ from 2 letters

$$b\ \boldsymbol{a}\ b\ \boldsymbol{a}\ \boldsymbol{a} \quad a\ \boldsymbol{b}\ a\ a\ \boldsymbol{b}$$

Hence $W_2(2k) = \binom{2k}{k}$.

With $k = \frac{1}{2}$: $\binom{1}{1/2} = \frac{1!}{(1/2)!^2} = \frac{1}{\Gamma^2(3/2)} = \frac{4}{\pi}$

EG $W_2(2k)$: abelian squares of length $2k$ from 2 letters

$$b\ \boldsymbol{a}\ b\ \boldsymbol{a}\ \boldsymbol{a} \quad a\ \boldsymbol{b}\ a\ a\ \boldsymbol{b}$$

Hence $W_2(2k) = \binom{2k}{k}$.

With $k = \frac{1}{2}$: $\binom{1}{1/2} = \frac{1!}{(1/2)!^2} = \frac{1}{\Gamma^2(3/2)} = \frac{4}{\pi}$

THM Carlson If $f(z)$ is analytic for $\mathrm{Re}\,(z) \geqslant 0$, "nice", and

$$f(0) = 0, \quad f(1) = 0, \quad f(2) = 0, \quad \ldots,$$

then $f(z) = 0$ identically.

EG $W_2(2k)$: abelian squares of length $2k$ from 2 letters

$$b\ \boldsymbol{a}\ b\ \boldsymbol{a}\ \boldsymbol{a} \quad a\ \boldsymbol{b}\ a\ a\ \boldsymbol{b}$$

Hence $W_2(2k) = \binom{2k}{k}$.

With $k = \frac{1}{2}$: $\binom{1}{1/2} = \frac{1!}{(1/2)!^2} = \frac{1}{\Gamma^2(3/2)} = \frac{4}{\pi}$

THM Carlson If $f(z)$ is analytic for $\mathrm{Re}\,(z) \geqslant 0$, "nice", and

$$f(0) = 0, \quad f(1) = 0, \quad f(2) = 0, \quad \ldots,$$

then $f(z) = 0$ identically.

$|f(z)| \leqslant Ae^{\alpha|z|}$, and

$|f(iy)| \leqslant Be^{\beta|y|}$ for $\beta < \pi$

EG $W_2(2k)$: abelian squares of length $2k$ from 2 letters

$$b\ \boldsymbol{a}\ b\ \boldsymbol{a}\ \boldsymbol{a} \quad a\ \boldsymbol{b}\ a\ a\ \boldsymbol{b}$$

Hence $W_2(2k) = \binom{2k}{k}$.

With $k = \frac{1}{2}$: $\binom{1}{1/2} = \frac{1!}{(1/2)!^2} = \frac{1}{\Gamma^2(3/2)} = \frac{4}{\pi}$

THM Carlson If $f(z)$ is analytic for $\mathrm{Re}\,(z) \geqslant 0$, "nice", and

$$f(0) = 0, \quad f(1) = 0, \quad f(2) = 0, \quad \ldots,$$

then $f(z) = 0$ identically.

- $W_n(s)$ is nice!

"nice": $|f(z)| \leqslant Ae^{\alpha|z|}$, and $|f(iy)| \leqslant Be^{\beta|y|}$ for $\beta < \pi$

EG $W_2(2k)$: abelian squares of length $2k$ from 2 letters

$$b\ \boldsymbol{a}\ b\ \boldsymbol{a}\ \boldsymbol{a} \quad a\ \boldsymbol{b}\ a\ a\ \boldsymbol{b}$$

Hence $W_2(2k) = \binom{2k}{k}$.

With $k = \frac{1}{2}$: $\binom{1}{1/2} = \frac{1!}{(1/2)!^2} = \frac{1}{\Gamma^2(3/2)} = \frac{4}{\pi}$

THM Carlson

If $f(z)$ is analytic for $\mathrm{Re}\,(z) \geqslant 0$, "nice", and

$$f(0) = 0, \quad f(1) = 0, \quad f(2) = 0, \quad \ldots,$$

then $f(z) = 0$ identically.

- $W_n(s)$ is nice!
- Indeed, $W_2(s) = \binom{s}{s/2}$.

("nice" →) $|f(z)| \leqslant Ae^{\alpha|z|}$, and
$|f(iy)| \leqslant Be^{\beta|y|}$ for $\beta < \pi$

Moments of a three-step walk

EG

$$W_3(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} = {}_3F_2\left(\left.\begin{matrix} \frac{1}{2}, -k, -k \\ 1, 1 \end{matrix}\right| 4\right)$$

Moments of a three-step walk

EG

$$W_3(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} = {}_3F_2\left(\left.\begin{matrix}\frac{1}{2}, -k, -k\\ 1, 1\end{matrix}\right| 4\right)$$

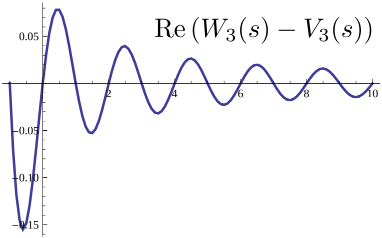
$${}_3F_2\left(\left.\begin{matrix}\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\\ 1, 1\end{matrix}\right| 4\right) \approx 1.574597238 - 0.126026522i$$

Moments of a three-step walk

EG

$$W_3(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} = \underbrace{{}_3F_2\left(\begin{matrix}\frac{1}{2}, -k, -k \\ 1, 1\end{matrix}\middle| 4\right)}_{=:V_3(2k)}$$

$${}_3F_2\left(\begin{matrix}\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \\ 1, 1\end{matrix}\middle| 4\right) \approx 1.574597238 - 0.126026522i$$

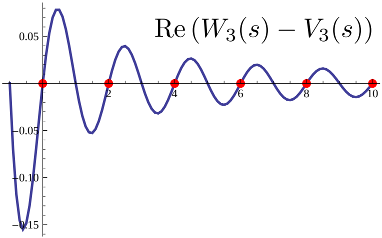


Moments of a three-step walk

EG

$$W_3(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} = \underbrace{{}_3F_2\left(\begin{matrix}\frac{1}{2}, -k, -k \\ 1, 1\end{matrix}\middle| 4\right)}_{=:V_3(2k)}$$

$$ {}_3F_2\left(\begin{matrix}\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \\ 1, 1\end{matrix}\middle| 4\right) \approx 1.574597238 - 0.126026522i$$

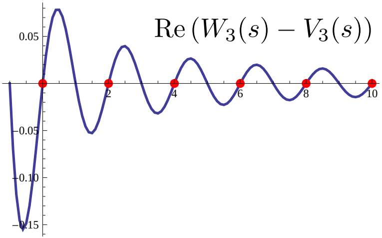


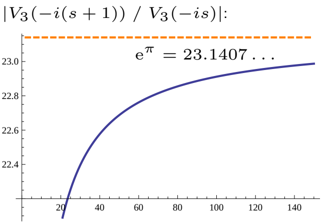
Moments of a three-step walk

EG

$$W_3(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} = \underbrace{{}_3F_2\left(\begin{matrix}\frac{1}{2}, -k, -k\\ 1, 1\end{matrix}\middle| 4\right)}_{=:V_3(2k)}$$

$${}_3F_2\left(\begin{matrix}\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\\ 1, 1\end{matrix}\middle| 4\right) \approx 1.574597238 - 0.126026522i$$





EG

$$W_3(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} = \underbrace{{}_3F_2\left(\left.\begin{matrix} \frac{1}{2}, -k, -k \\ 1, 1 \end{matrix}\right| 4\right)}_{=: V_3(2k)}$$

THM Borwein-Nuyens-S-Wan, 2010

For integers k,

$$W_3(k) = \operatorname{Re} {}_3F_2\left(\left.\begin{matrix} \frac{1}{2}, -\frac{k}{2}, -\frac{k}{2} \\ 1, 1 \end{matrix}\right| 4\right).$$

Moments of a three-step walk

EG

$$W_3(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} = \underbrace{{}_3F_2\left(\left.\begin{matrix}\frac{1}{2}, -k, -k\\ 1, 1\end{matrix}\right| 4\right)}_{=:V_3(2k)}$$

THM Borwein-Nuyens-S-Wan, 2010

For integers k,

$$W_3(k) = \operatorname{Re}\, {}_3F_2\left(\left.\begin{matrix}\frac{1}{2}, -\frac{k}{2}, -\frac{k}{2}\\ 1, 1\end{matrix}\right| 4\right).$$

COR

$$W_3(1) = \frac{3}{16}\frac{2^{1/3}}{\pi^4}\Gamma^6\left(\frac{1}{3}\right) + \frac{27}{4}\frac{2^{2/3}}{\pi^4}\Gamma^6\left(\frac{2}{3}\right)$$
$$= 1.57459723755189\ldots$$

Moments of a four-step walk

- Using Meijer G-function representations and transformations:

THM Borwein-S-Wan, 2010

$$W_4(-1) = \frac{\pi}{4}\, {}_7F_6\left(\left.\begin{matrix} \frac{5}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{4}, 1, 1, 1, 1, 1 \end{matrix}\right| 1\right)$$

$$= \frac{\pi}{4} {}_6F_5\left(\left.\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1, 1, 1 \end{matrix}\right| 1\right) + \frac{\pi}{64} {}_6F_5\left(\left.\begin{matrix} \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \\ 2, 2, 2, 2, 2 \end{matrix}\right| 1\right)$$

$$= \frac{\pi}{4} \sum_{n=0}^{\infty} \frac{(4n+1)\binom{2n}{n}^6}{4^{6n}}.$$

THM Borwein-S-Wan, 2010

$$W_4(1) = \frac{3\pi}{4} {}_7F_6\left(\left.\begin{matrix} \frac{7}{4}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{4}, 2, 2, 2, 1, 1 \end{matrix}\right| 1\right)$$

$$- \frac{3\pi}{8} {}_7F_6\left(\left.\begin{matrix} \frac{7}{4}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{4}, 2, 2, 2, 2, 1 \end{matrix}\right| 1\right).$$

- We have no idea about the case of five steps.

A combinatorial convolution

- From the interpretation as abelian squares:

$$W_{n+m}(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 W_n(2j)\, W_m(2(k-j)).$$

A combinatorial convolution

- From the interpretation as abelian squares:

$$W_{n+m}(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 W_n(2j)\, W_m(2(k-j)).$$

CONJ For even n,

$$W_n(s) \stackrel{?}{=} \sum_{j=0}^{\infty} \binom{s/2}{j}^2 W_{n-1}(s-2j).$$

A combinatorial convolution

- From the interpretation as abelian squares:

$$W_{n+m}(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 W_n(2j)\, W_m(2(k-j)).$$

CONJ For even n,

$$W_n(s) \overset{?}{=} \sum_{j=0}^{\infty} \binom{s/2}{j}^2 W_{n-1}(s-2j).$$

- True for even s
- True for $n = 2$
- True for $n = 4$ and integer s
- In general, proven up to some technical growth conditions

Complex moments

THM

$$W_n(2k) = \sum_{a_1+\cdots+a_n=k} \binom{k}{a_1,\ldots,a_n}^2$$

- Inevitable **recursions**

$K \cdot f(k) = f(k+1)$

$$\left[(k+2)^2 K^2 - (10k^2+30k+23)K + 9(k+1)^2\right] \cdot W_3(2k) = 0$$

$$\left[(k+2)^3 K^2 - (2k+3)(10k^2+30k+24)K + 64(k+1)^3\right] \cdot W_4(2k) = 0$$

Complex moments

THM

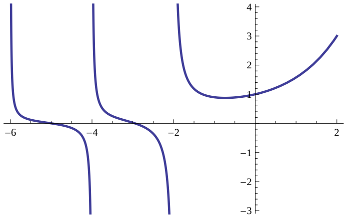
$$W_n(2k) = \sum_{a_1+\cdots+a_n=k} \binom{k}{a_1,\ldots,a_n}^2$$

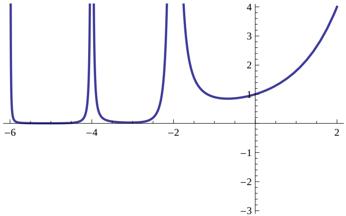
- Inevitable **recursions** $K \cdot f(k) = f(k+1)$

$$\left[(k+2)^2K^2 - (10k^2+30k+23)K + 9(k+1)^2\right] \cdot W_3(2k) = 0$$

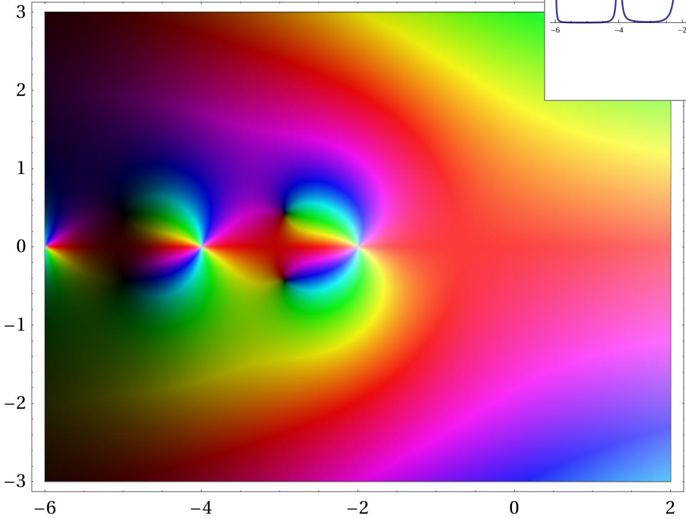
$$\left[(k+2)^3K^2 - (2k+3)(10k^2+30k+24)K + 64(k+1)^3\right] \cdot W_4(2k) = 0$$

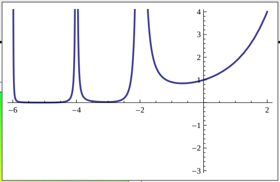
- Via **Carlson's Theorem** these become functional equations
- $W_3(s)$ has a simple pole at -2 with residue $\frac{2}{\sqrt{3}\pi}$; others at $-2k$.



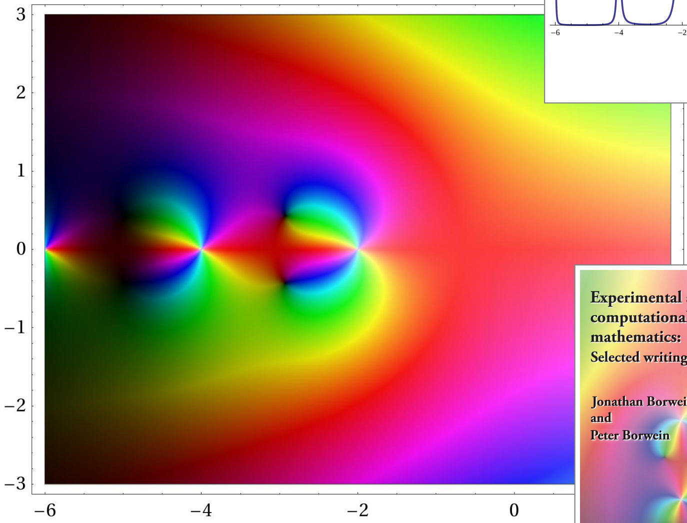


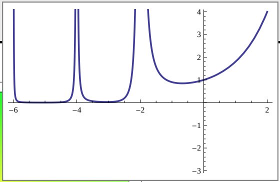
$W_4(s)$ in the complex plane

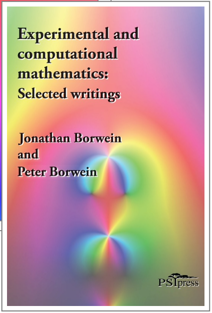




$W_4(s)$ in the complex plane







Crashcourse on the Mellin transform

- Mellin transform $F(s)$ of $f(x)$:
  $\mathcal{M}\left[f;s\right] = \int_0^\infty x^s f(x) \frac{\mathrm{d}x}{x}$

$$W_n(s-1) = \mathcal{M}\left[p_n;s\right]$$

Crashcourse on the Mellin transform

- Mellin transform $F(s)$ of $f(x)$:
  $$\mathcal{M}[f;s] = \int_0^\infty x^s f(x) \frac{\mathrm{d}x}{x}$$
- $F(s)$ is analytic in a strip
- Functional properties:
  - $\mathcal{M}[x^\mu f(x);s] = F(s+\mu)$
  - $\mathcal{M}[D_x f(x);s] = -(s-1)F(s-1)$
  - $\mathcal{M}[-\theta_x f(x);s] = sF(s)$

$$W_n(s-1) = \mathcal{M}[p_n;s]$$

Thus functional equations for $F(s)$ translate into DEs for $f(x)$

Crashcourse on the Mellin transform

- Mellin transform $F(s)$ of $f(x)$:
  $$\mathcal{M}[f;s] = \int_0^\infty x^s f(x) \frac{\mathrm{d}x}{x}$$
- $F(s)$ is analytic in a strip
- Functional properties:
  - $\mathcal{M}[x^\mu f(x); s] = F(s+\mu)$
  - $\mathcal{M}[D_x f(x); s] = -(s-1)F(s-1)$
  - $\mathcal{M}[-\theta_x f(x); s] = sF(s)$

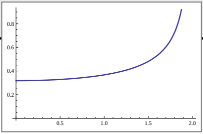
$$W_n(s-1) = \mathcal{M}[p_n; s]$$

Thus functional equations for $F(s)$ translate into DEs for $f(x)$

- Poles of $F(s)$ left of strip $\Longrightarrow$ asymptotics of $f(x)$ at zero

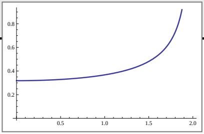
  $\frac{1}{(s+m)^{n+1}}$ $\qquad$ $\frac{(-1)^n}{n!} x^m (\log x)^n$

Mellin approach illustrated for p_2



- $W_2(2k) = \binom{2k}{k}$

$$(s+2)W_2(s+2) - 4(s+1)W_2(s) = 0$$
$$\left[x^2\left(\theta_x+1\right) - 4\theta_x\right] \cdot p_2(x) = 0$$

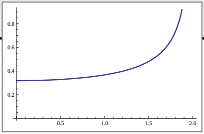


- $W_2(2k) = \binom{2k}{k}$

$$(s+2)W_2(s+2) - 4(s+1)W_2(s) = 0$$
$$\left[x^2(\theta_x + 1) - 4\theta_x\right] \cdot p_2(x) = 0$$

- Hence: $p_2(x) = \frac{C}{\sqrt{4-x^2}}$

Mellin approach illustrated for p_2



- $W_2(2k) = \binom{2k}{k}$

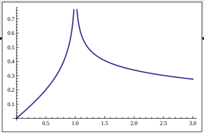
$$(s+2)W_2(s+2) - 4(s+1)W_2(s) = 0$$
$$\left[x^2(\theta_x+1) - 4\theta_x\right] \cdot p_2(x) = 0$$

- Hence: $p_2(x) = \frac{C}{\sqrt{4-x^2}}$

$$W_2(s) = \frac{1}{\pi}\frac{1}{s+1} + O(1) \text{ as } s \to -1$$
$$p_2(x) = \frac{1}{\pi} + O(x) \text{ as } x \to 0^+$$

- Taken together: $p_2(x) = \frac{2}{\pi\sqrt{4-x^2}}$

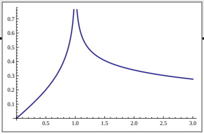
p_3 in hypergeometric form



- $W_3(s)$ has simple poles at $-2k-2$ with residue

$$\frac{2}{\pi\sqrt{3}}\frac{W_3(2k)}{3^{2k}}$$

$$p_3(x) = \frac{2x}{\pi\sqrt{3}} \sum_{k=0}^{\infty} W_3(2k) \left(\frac{x}{3}\right)^{2k} \qquad \text{for } 0 \leqslant x \leqslant 1$$



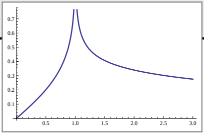
- $W_3(s)$ has simple poles at $-2k-2$ with residue

$$\frac{2}{\pi\sqrt{3}} \frac{W_3(2k)}{3^{2k}}$$

$$p_3(x) = \frac{2x}{\pi\sqrt{3}} \sum_{k=0}^{\infty} W_3(2k) \left(\frac{x}{3}\right)^{2k} \qquad \text{for } 0 \leqslant x \leqslant 1$$

- $W_3(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j}$ is an **Apéry-like** sequence

p_3 in hypergeometric form



- $W_3(s)$ has simple poles at $-2k-2$ with residue

$$\frac{2}{\pi\sqrt{3}} \frac{W_3(2k)}{3^{2k}}$$

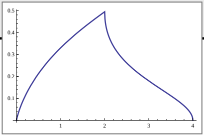
$$p_3(x) = \frac{2x}{\pi\sqrt{3}} \sum_{k=0}^{\infty} W_3(2k) \left(\frac{x}{3}\right)^{2k} \qquad \text{for } 0 \leqslant x \leqslant 1$$

- $W_3(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j}$ is an **Apéry-like** sequence

$$p_3(x) = \frac{2\sqrt{3}x}{\pi(3+x^2)} \, {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \frac{x^2(9-x^2)^2}{(3+x^2)^3}\right)$$

- Easy to verify once found
- Holds for $0 \leqslant x \leqslant 3$

p_4 and its differential equation

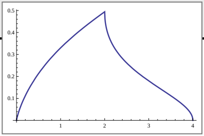


$$\left[(s+4)^3 S^4 - 4(s+3)(5s^2+30s+48)S^2 + 64(s+2)^3\right] \cdot W_4(s) = 0$$

translates into $A_4 \cdot p_4(x) = 0$ with

$$A_4 = x^4(\theta_x+1)^3 - 4x^2\theta_x(5\theta_x^2+3) + 64(\theta_x-1)^3$$

p_4 and its differential equation



$$\left[(s+4)^3 S^4 - 4(s+3)(5s^2+30s+48)S^2 + 64(s+2)^3\right] \cdot W_4(s) = 0$$

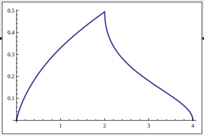
translates into $A_4 \cdot p_4(x) = 0$ with

$$A_4 = x^4(\theta_x+1)^3 - 4x^2\theta_x(5\theta_x^2+3) + 64(\theta_x-1)^3$$

!! Care needed

$p_4(x) \approx C\sqrt{4-x}$ as $x \to 4^-$. Thus p_4'' is not locally integrable and does not have a Mellin transform in the classical sense.

p_4 and its differential equation



$$\left[(s+4)^3 S^4 - 4(s+3)(5s^2+30s+48)S^2 + 64(s+2)^3\right] \cdot W_4(s) = 0$$

translates into $A_4 \cdot p_4(x) = 0$ with

$$\begin{aligned} A_4 &= x^4(\theta_x+1)^3 - 4x^2\theta_x(5\theta_x^2+3) + 64(\theta_x-1)^3 \\ &= (x-4)(x-2)x^3(x+2)(x+4)D_x^3 + 6x^4\left(x^2-10\right)D_x^2 \\ &\quad + x\left(7x^4-32x^2+64\right)D_x + \left(x^2-8\right)\left(x^2+8\right) \end{aligned}$$

!! **Care needed**

$p_4(x) \approx C\sqrt{4-x}$ as $x \to 4^-$. Thus p_4'' is not locally integrable and does not have a Mellin transform in the classical sense.

Densities in general

THM Borwein-S-Wan-Zudilin, 2011

- The density p_n satisfies a DE of order $n-1$.
- p_n is real analytic except at 0 and the integers $n, n-2, n-4, \ldots$.

THM Borwein-S-Wan-Zudilin, 2011

- The density p_n satisfies a DE of order $n-1$.
- p_n is real analytic except at 0 and the integers $n, n-2, n-4, \ldots$.

The second statement relies on an explicit recursion by Verrill (2004) as well as the combinatorial identity

$$\sum_{\substack{0 \leqslant m_1, \ldots, m_j < n/2 \\ m_i < m_{i+1}}} \prod_{i=1}^{j} (n-2m_i)^2 = \sum_{\substack{1 \leqslant \alpha_1, \ldots, \alpha_j \leqslant n \\ \alpha_i \leqslant \alpha_{i+1}-2}} \prod_{i=1}^{j} \alpha_i (n+1-\alpha_i).$$

First proven by Djakov-Mityagin (2004).

Direct combinatorial proof by Zagier.

Densities in general

EG

$$\sum_{m=0}^{n/2-1}(n-2m)^2=\sum_{\alpha=1}^{n}\alpha(n+1-\alpha)=\binom{n+2}{3}$$

$$\sum_{m_1=0}^{n/2-1}\sum_{m_2=0}^{m_1-1}(n-2m_1)^2(n-2m_2)^2=\sum_{\alpha_1=1}^{n}\sum_{\alpha_2=1}^{\alpha_1-2}\alpha_1(n+1-\alpha_1)\alpha_2(n+1-\alpha_2)$$

$$\sum_{\substack{0\leqslant m_1,\ldots,m_j<n/2\\ m_i<m_{i+1}}}\prod_{i=1}^{j}(n-2m_i)^2=\sum_{\substack{1\leqslant\alpha_1,\ldots,\alpha_j\leqslant n\\ \alpha_i\leqslant\alpha_{i+1}-2}}\prod_{i=1}^{j}\alpha_i(n+1-\alpha_i).$$

First proven by Djakov-Mityagin (2004).

Direct combinatorial proof by Zagier.

p_4 and its asymptotics at zero

EG

$$W_4(s) = \frac{3}{2\pi^2}\frac{1}{(s+2)^2} + \frac{9\log 2}{2\pi^2}\frac{1}{s+2} + O(1) \quad \text{as } s \to -2$$

$$p_4(x) = -\frac{3}{2\pi^2}x\log(x) + \frac{9\log 2}{2\pi^2}x + O(x^3) \quad \text{as } x \to 0^+$$

- $W_4(s)$ has double poles:

$$W_4(s) = \frac{s_{4,k}}{(s+2k+2)^2} + \frac{r_{4,k}}{s+2k+2} + O(1) \quad \text{as } s \to -2k-2$$

p_4 and its asymptotics at zero

EG

$$W_4(s) = \frac{3}{2\pi^2}\frac{1}{(s+2)^2} + \frac{9\log 2}{2\pi^2}\frac{1}{s+2} + O(1) \quad \text{as } s \to -2$$

$$p_4(x) = -\frac{3}{2\pi^2} x \log(x) + \frac{9\log 2}{2\pi^2} x + O(x^3) \quad \text{as } x \to 0^+$$

- $W_4(s)$ has double poles:

$$W_4(s) = \frac{s_{4,k}}{(s+2k+2)^2} + \frac{r_{4,k}}{s+2k+2} + O(1) \quad \text{as } s \to -2k-2$$

$$p_4(x) = \sum_{k=0}^{\infty} (r_{4,k} - s_{4,k}\log(x))\, x^{2k+1} \qquad \text{for small } x \geqslant 0$$

EG

$$W_4(s) = \frac{3}{2\pi^2}\frac{1}{(s+2)^2} + \frac{9\log 2}{2\pi^2}\frac{1}{s+2} + O(1) \quad \text{as } s \to -2$$

$$p_4(x) = -\frac{3}{2\pi^2} x \log(x) + \frac{9\log 2}{2\pi^2} x + O(x^3) \quad \text{as } x \to 0^+$$

- $W_4(s)$ has double poles:

$$W_4(s) = \frac{s_{4,k}}{(s+2k+2)^2} + \frac{r_{4,k}}{s+2k+2} + O(1) \quad \text{as } s \to -2k-2$$

$$p_4(x) = \sum_{k=0}^{\infty} (r_{4,k} - s_{4,k}\log(x))\, x^{2k+1} \quad \text{for small } x \geqslant 0$$

☺☺

$$s_{4,k} = \frac{3}{2\pi^2}\frac{W_4(2k)}{8^{2k}} \qquad W_4(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} \binom{2n-2j}{n-j}$$

$r_{4,k}$ known recursively

Domb numbers

The Domb numbers

- $y_0(z) := \sum_{k\geqslant 0} W_4(2k) z^k$ is the analytic solution of

$$\left[64z^2(\theta+1)^3 - 2z(2\theta+1)(5\theta^2+5\theta+2) + \theta^3\right] \cdot y(z) = 0. \quad \text{(DE)}$$

- Let $y_1(z)$ solve (DE) and $y_1(z) - y_0(z)\log(z) \in z\mathbb{Q}[[z]]$.
  Then $p_4(x) = -\frac{3x}{4\pi^2}\, y_1(x^2/64)$.

The Domb numbers

- $y_0(z) := \sum_{k\geqslant 0} W_4(2k) z^k$ is the analytic solution of

$$\left[64z^2(\theta+1)^3 - 2z(2\theta+1)(5\theta^2+5\theta+2) + \theta^3\right] \cdot y(z) = 0. \quad \text{(DE)}$$

- Let $y_1(z)$ solve (DE) and $y_1(z) - y_0(z)\log(z) \in z\mathbb{Q}[[z]]$.
  Then $p_4(x) = -\frac{3x}{4\pi^2}\, y_1(x^2/64)$.

THM Chan-Chan-Liu 2004; Rogers 2009

Generating function for Domb numbers:

$$\sum_{k=0}^{\infty} W_4(2k) z^k = \frac{1}{1-4z}\, {}_3F_2\left(\begin{matrix}\frac{1}{3}, \frac{1}{2}, \frac{2}{3} \\ 1, 1\end{matrix} \middle| \frac{108z^2}{(1-4z)^3}\right)$$

The Domb numbers

- $y_0(z) := \sum_{k\geqslant 0} W_4(2k) z^k$ is the analytic solution of

$$\left[64z^2(\theta+1)^3 - 2z(2\theta+1)(5\theta^2+5\theta+2) + \theta^3\right] \cdot y(z) = 0. \quad \text{(DE)}$$

- Let $y_1(z)$ solve (DE) and $y_1(z) - y_0(z)\log(z) \in z\mathbb{Q}[[z]]$.
  Then $p_4(x) = -\frac{3x}{4\pi^2}\, y_1(x^2/64)$.

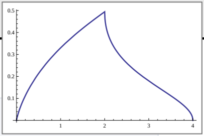
THM Chan-Chan-Liu 2004; Rogers 2009

Generating function for Domb numbers:

$$\sum_{k=0}^{\infty} W_4(2k) z^k = \frac{1}{1-4z}\, {}_3F_2\left(\left.\begin{matrix}\frac{1}{3}, \frac{1}{2}, \frac{2}{3} \\ 1, 1\end{matrix}\right| \frac{108z^2}{(1-4z)^3}\right)$$

- Basis at ∞ for the hypergeometric equation of ${}_3F_2\left(\left.\begin{matrix}\frac{1}{3},\frac{1}{2},\frac{2}{3} \\ 1,1\end{matrix}\right| t\right)$:
  [as $x \to 4$ then $z = \frac{x^2}{64} \to \frac{1}{4}$ and $t = \frac{108z^2}{(1-4z)^3} \to \infty$]

$$t^{-1/3}{}_3F_2\left(\left.\begin{matrix}\frac{1}{3},\frac{1}{3},\frac{1}{3} \\ \frac{2}{3},\frac{5}{6}\end{matrix}\right| \frac{1}{t}\right), \quad t^{-1/2}{}_3F_2\left(\left.\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2} \\ \frac{5}{6},\frac{7}{6}\end{matrix}\right| \frac{1}{t}\right), \quad t^{-2/3}{}_3F_2\left(\left.\begin{matrix}\frac{2}{3},\frac{2}{3},\frac{2}{3} \\ \frac{4}{3},\frac{7}{6}\end{matrix}\right| \frac{1}{t}\right)$$



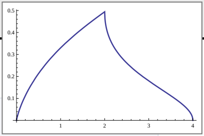
THM Borwein-S-Wan-Zudilin 2011

For $2 \leqslant x \leqslant 4$,

$$p_4(x) = \frac{2}{\pi^2} \frac{\sqrt{16-x^2}}{x} \, {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{5}{6}, \frac{7}{6} \end{matrix} \middle| \frac{(16-x^2)^3}{108x^4} \right).$$

- Easily (if tediously) provable once found

p_4 in hypergeometric form



THM Borwein-S-Wan-Zudilin 2011

For $2 \leqslant x \leqslant 4$,

$$p_4(x) = \frac{2}{\pi^2} \frac{\sqrt{16-x^2}}{x} \, {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{5}{6}, \frac{7}{6} \end{matrix} \middle| \frac{(16-x^2)^3}{108x^4} \right).$$

- Easily (if tediously) provable once found
- Quite marvelously, as first observed numerically:

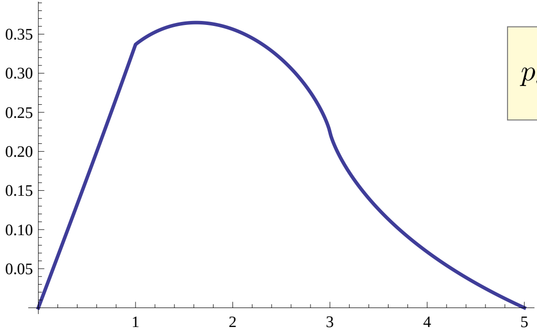
THM Borwein-S-Wan-Zudilin 2011

For $0 \leqslant x \leqslant 4$,

$$p_4(x) = \frac{2}{\pi^2} \frac{\sqrt{16-x^2}}{x} \, \mathrm{Re} \, {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{5}{6}, \frac{7}{6} \end{matrix} \middle| \frac{(16-x^2)^3}{108x^4} \right).$$

The density of a five-step random walk, again

$$p_5(x) = 0.32993\,x + 0.0066167x^3 + 0.00026233x^5 + 0.000014119x^7 + O(x^9)$$

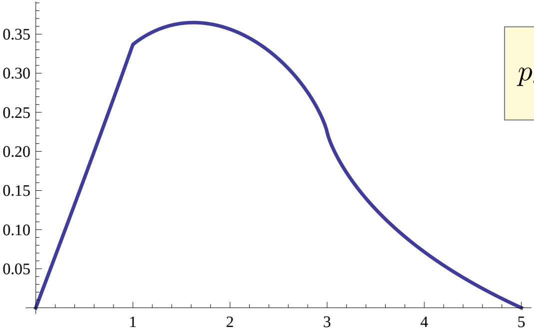


$$p_5(x) = \int_0^\infty xt J_0(xt) J_0^5(t)\,\mathrm{d}t$$

> *. . . the graphical construction, however carefully reinvestigated, did not permit of our considering the curve to be anything but a* **straight** *line. . . Even if it is not absolutely true, it exemplifies the extraordinary power of such integrals of J products to give extremely close approximations to such simple forms as horizontal lines.*
>
> **Karl Pearson**, *1906*

The density of a five-step random walk, again

$$p_5(x) = \underbrace{0.32993}_{=p_4(1)}\,x + 0.0066167x^3 + 0.00026233x^5 + 0.000014119x^7 + O(x^9)$$



$$p_5(x) = \int_0^\infty xtJ_0(xt)J_0^5(t)\,\mathrm{d}t$$

> *. . . the graphical construction, however carefully reinvestigated, did not permit of our considering the curve to be anything but a* **straight** *line. . . Even if it is not absolutely true, it exemplifies the extraordinary power of such integrals of J products to give extremely close approximations to such simple forms as horizontal lines.*
>
> **Karl Pearson**, *1906*

Modular differential equations

THM Let $f(\tau)$ be a modular form and $x(\tau)$ a modular function w.r.t. Γ.

- Then $y(x)$ defined by $f(\tau) = y(x(\tau))$ satisfies a linear DE.
- If $x(\tau)$ is a Hauptmodul for Γ, then the DE has polynomial coefficients.
- The solutions of the DE are $y(x), \tau y(x), \tau^2 y(x), \ldots$.

Modular differential equations

THM Let $f(\tau)$ be a modular form and $x(\tau)$ a modular function w.r.t. Γ.

- Then $y(x)$ defined by $f(\tau) = y(x(\tau))$ satisfies a linear DE.
- If $x(\tau)$ is a Hauptmodul for Γ, then the DE has polynomial coefficients.
- The solutions of the DE are $y(x), \tau y(x), \tau^2 y(x), \ldots$.

- Dedekind eta function: $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$, $\qquad q = e^{2\pi i\tau}$

EG
Classic

$$ {}_2F_1 \left(\begin{matrix} 1/2, 1/2 \\ 1 \end{matrix} \middle| \lambda(\tau) \right) = \theta_3(\tau)^2 $$

- $\lambda(\tau) = 16 \frac{\eta(\tau/2)^8 \eta(2\tau)^{16}}{\eta(\tau)^{24}}$ is the elliptic lambda function, a Hauptmodul for $\Gamma(2)$.
- $\theta_3(\tau) = \frac{\eta(\tau)^5}{\eta(\tau/2)^2 \eta(2\tau)^2}$ is the usual Jacobi theta function.

Modular differential equations

THM Let $f(\tau)$ be a modular form and $x(\tau)$ a modular function w.r.t. Γ.

- Then $y(x)$ defined by $f(\tau) = y(x(\tau))$ satisfies a linear DE.
- If $x(\tau)$ is a Hauptmodul for Γ, then the DE has polynomial coefficients.
- The solutions of the DE are $y(x), \tau y(x), \tau^2 y(x), \ldots$.

- Dedekind eta function: $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$, $\quad q = e^{2\pi i \tau}$

EG Chan-Chan-Liu 2004

$$x(\tau) = -\left(\frac{\eta(2\tau)\eta(6\tau)}{\eta(\tau)\eta(3\tau)}\right)^6 = -q - 6q^2 - 21q^3 - 68q^4 + \ldots, \qquad f(\tau) = \frac{(\eta(\tau)\eta(3\tau))^4}{(\eta(2\tau)\eta(6\tau))^2} = 1 - 4q + 4q^2 - 4q^3 + 20q^4 + \ldots$$

Here, $\Gamma = \left\langle \Gamma_0(6), \frac{1}{\sqrt{3}} \left(\begin{smallmatrix} 3 & -2 \\ 6 & -3 \end{smallmatrix}\right) \right\rangle$.

Modular differential equations

THM Let $f(\tau)$ be a modular form and $x(\tau)$ a modular function w.r.t. Γ.

- Then $y(x)$ defined by $f(\tau) = y(x(\tau))$ satisfies a linear DE.
- If $x(\tau)$ is a Hauptmodul for Γ, then the DE has polynomial coefficients.
- The solutions of the DE are $y(x), \tau y(x), \tau^2 y(x), \ldots$.

- Dedekind eta function: $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$, $\qquad q = e^{2\pi i \tau}$

EG Chan-Chan-Liu 2004

$$x(\tau) = -\left(\frac{\eta(2\tau)\eta(6\tau)}{\eta(\tau)\eta(3\tau)}\right)^6 = -q - 6q^2 - 21q^3 - 68q^4 + \ldots, \qquad f(\tau) = \frac{(\eta(\tau)\eta(3\tau))^4}{(\eta(2\tau)\eta(6\tau))^2} = 1 - 4q + 4q^2 - 4q^3 + 20q^4 + \ldots$$

Here, $\Gamma = \left\langle \Gamma_0(6), \frac{1}{\sqrt{3}} \left(\begin{smallmatrix} 3 & -2 \\ 6 & -3 \end{smallmatrix}\right) \right\rangle$. Then, in a neighborhood of $i\infty$,

$$f(\tau) = y_0(x(\tau)) = \sum_{k=0}^{\infty} W_4(2k) x(\tau)^k.$$

Modular parametrization of p_4

THM Borwein-S-Wan-Zudilin 2011

For $\tau = -1/2 + iy$ and $y > 0$:

$$p_4\Bigg(\underbrace{8i\left(\frac{\eta(2\tau)\eta(6\tau)}{\eta(\tau)\eta(3\tau)}\right)^3}_{=\sqrt{64x(\tau)}}\Bigg) = \frac{6(2\tau+1)}{\pi}\underbrace{\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)}_{=\sqrt{-x(\tau)}f(\tau)}$$

Modular parametrization of p_4

THM Borwein-S-Wan-Zudilin 2011

For $\tau = -1/2 + iy$ and $y > 0$:

$$p_4\Bigg(\underbrace{8i\left(\frac{\eta(2\tau)\eta(6\tau)}{\eta(\tau)\eta(3\tau)}\right)^3}_{=\sqrt{64x(\tau)}}\Bigg) = \frac{6(2\tau+1)}{\pi}\underbrace{\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)}_{=\sqrt{-x(\tau)}f(\tau)}$$

- When $\tau = -\frac{1}{2} + \frac{1}{6}\sqrt{-15}$, one obtains $p_4(1) = p_5'(0)$ as an η-product.

Modular parametrization of p_4

THM Borwein-S-Wan-Zudilin 2011

For $\tau = -1/2 + iy$ and $y > 0$:

$$p_4\Bigg(\underbrace{8i\left(\frac{\eta(2\tau)\eta(6\tau)}{\eta(\tau)\eta(3\tau)}\right)^3}_{=\sqrt{64x(\tau)}}\Bigg) = \frac{6(2\tau+1)}{\pi}\underbrace{\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)}_{=\sqrt{-x(\tau)}f(\tau)}$$

- When $\tau = -\frac{1}{2} + \frac{1}{6}\sqrt{-15}$, one obtains $p_4(1) = p_5'(0)$ as an η-product.
- Applying the Chowla–Selberg formula, eventually leads to:

COR

$$p_4(1) = p_5'(0) = \frac{\sqrt{5}}{40\pi^4}\Gamma(\tfrac{1}{15})\Gamma(\tfrac{2}{15})\Gamma(\tfrac{4}{15})\Gamma(\tfrac{8}{15}) \approx 0.32993$$

Chowla–Selberg formula

THM Chowla–Selberg 1967

$$\prod_{j=1}^{h} a_j^{-6} |\eta(\tau_j)|^{24} = \frac{1}{(2\pi|d|)^{6h}} \left[\prod_{k=1}^{|d|} \Gamma\left(\frac{k}{|d|}\right)^{\left(\frac{d}{k}\right)} \right]^{3w}$$

where the product is over reduced binary quadratic forms $[a_j, b_j, c_j]$ of discriminant $d < 0$. Further, $\tau_j = \frac{-b_j + \sqrt{d}}{2a_j}$.

Chowla–Selberg formula

THM Chowla–Selberg 1967

$$\prod_{j=1}^{h} a_j^{-6}|\eta(\tau_j)|^{24} = \frac{1}{(2\pi|d|)^{6h}} \left[\prod_{k=1}^{|d|} \Gamma\left(\frac{k}{|d|}\right)^{\left(\frac{d}{k}\right)} \right]^{3w}$$

where the product is over reduced binary quadratic forms $[a_j, b_j, c_j]$ of discriminant $d < 0$. Further, $\tau_j = \frac{-b_j + \sqrt{d}}{2a_j}$.

EG $\mathbb{Q}(\sqrt{-15})$ has discriminant $\Delta = -15$ and class number $h = 2$.

$$Q_1 = [1, 1, 4], \qquad Q_2 = [2, 1, 2]$$

with corresponding roots

$$\tau_1 = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-15}, \qquad \tau_2 = \tfrac{1}{2}\tau_1.$$

$$\frac{1}{\sqrt{2}} |\eta(\tau_1)\eta(\tau_2)|^2 = \frac{1}{30\pi} \left(\frac{\Gamma(\frac{1}{15})\Gamma(\frac{2}{15})\Gamma(\frac{4}{15})\Gamma(\frac{8}{15})}{\Gamma(\frac{7}{15})\Gamma(\frac{11}{15})\Gamma(\frac{13}{15})\Gamma(\frac{14}{15})} \right)^{1/2}$$

$$= \frac{1}{120\pi^3} \Gamma(\tfrac{1}{15})\Gamma(\tfrac{2}{15})\Gamma(\tfrac{4}{15})\Gamma(\tfrac{8}{15})$$

Evaluating eta-quotients

Fact If $\sigma_1, \sigma_2 \in \mathcal{H}$ both belong to $\mathbb{Q}(\sqrt{-d})$, then the quotient $\eta(\sigma_1)/\eta(\sigma_2)$ is an algebraic number.

Evaluating eta-quotients

Fact If $\sigma_1, \sigma_2 \in \mathcal{H}$ both belong to $\mathbb{Q}(\sqrt{-d})$, then the quotient $\eta(\sigma_1)/\eta(\sigma_2)$ is an algebraic number.

Proof.

- We can write $\sigma_2 = M \cdot \sigma_1$ for some $M \in \mathrm{GL}_2(\mathbb{Z})$.
- $f(\tau) = \frac{\eta(\tau)}{\eta(M \cdot \tau)}$ is a modular function.

□

Evaluating eta-quotients

Fact If $\sigma_1, \sigma_2 \in \mathcal{H}$ both belong to $\mathbb{Q}(\sqrt{-d})$, then the quotient $\eta(\sigma_1)/\eta(\sigma_2)$ is an algebraic number.

Proof.

- We can write $\sigma_2 = M \cdot \sigma_1$ for some $M \in \mathrm{GL}_2(\mathbb{Z})$.
- $f(\tau) = \frac{\eta(\tau)}{\eta(M \cdot \tau)}$ is a modular function.
- $\sigma_1 = N \cdot \sigma_1$ for some non-identity $N \in \mathrm{GL}_2(\mathbb{Z})$.
- $f(N \cdot \tau)$ is another modular function.

□

Evaluating eta-quotients

Fact If $\sigma_1, \sigma_2 \in \mathcal{H}$ both belong to $\mathbb{Q}(\sqrt{-d})$, then the quotient $\eta(\sigma_1)/\eta(\sigma_2)$ is an algebraic number.

Proof.

- We can write $\sigma_2 = M \cdot \sigma_1$ for some $M \in \mathrm{GL}_2(\mathbb{Z})$.
- $f(\tau) = \frac{\eta(\tau)}{\eta(M \cdot \tau)}$ is a modular function.
- $\sigma_1 = N \cdot \sigma_1$ for some non-identity $N \in \mathrm{GL}_2(\mathbb{Z})$.
- $f(N \cdot \tau)$ is another modular function.
- There is an algebraic relation $\Phi(f(\tau), f(N \cdot \tau)) = 0$.

□

Evaluating eta-quotients

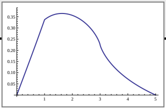
Fact If $\sigma_1, \sigma_2 \in \mathcal{H}$ both belong to $\mathbb{Q}(\sqrt{-d})$, then the quotient $\eta(\sigma_1)/\eta(\sigma_2)$ is an algebraic number.

Proof.

- We can write $\sigma_2 = M \cdot \sigma_1$ for some $M \in \mathrm{GL}_2(\mathbb{Z})$.
- $f(\tau) = \frac{\eta(\tau)}{\eta(M\cdot\tau)}$ is a modular function.
- $\sigma_1 = N \cdot \sigma_1$ for some non-identity $N \in \mathrm{GL}_2(\mathbb{Z})$.
- $f(N \cdot \tau)$ is another modular function.
- There is an algebraic relation $\Phi(f(\tau), f(N \cdot \tau)) = 0$.
- Then: $\Phi(f(\sigma_1), f(\sigma_1)) = 0$

□

What we know about p_5



- $W_5(s)$ has simple poles at $-2k-2$ with residue $r_{5,k}$
- Hence: $p_5(x) = \sum_{k=0}^{\infty} r_{5,k}\, x^{2k+1}$

THM Borwein-S-Wan-Zudilin, 2011

Surprising bonus of the modularity of p_4:

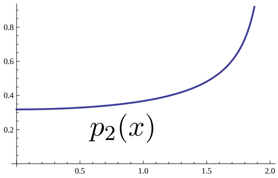
$$r_{5,0} = p_4(1) = \frac{\sqrt{5}}{40} \frac{\Gamma(\frac{1}{15})\Gamma(\frac{2}{15})\Gamma(\frac{4}{15})\Gamma(\frac{8}{15})}{\pi^4}$$

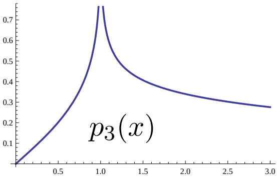
$$r_{5,1} \overset{?}{=} \frac{13}{225} r_{5,0} - \frac{2}{5\pi^4} \frac{1}{r_{5,0}}$$

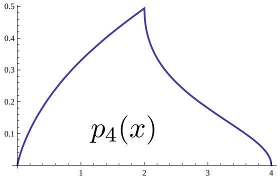
- Other residues given recursively
- p_5 solves the DE

$$\left[x^6(\theta+1)^4 - x^4(35\theta^4 + 42\theta^2 + 3) + x^2(259(\theta-1)^4 + 104(\theta-1)^2) - (15(\theta-3)(\theta-1))^2\right] \cdot p_5(x) = 0$$

Hypergeometric formulae summarized







$$p_2(x) = \frac{2}{\pi\sqrt{4-x^2}}$$

easy

$$p_3(x) = \frac{2\sqrt{3}}{\pi} \frac{x}{(3+x^2)} {}_2F_1\left(\begin{matrix} \frac{1}{3}, \frac{2}{3} \\ 1 \end{matrix} \middle| \frac{x^2\left(9-x^2\right)^2}{\left(3+x^2\right)^3}\right)$$

classical
with a spin

$$p_4(x) = \frac{2}{\pi^2} \frac{\sqrt{16-x^2}}{x} \operatorname{Re} {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{5}{6}, \frac{7}{6} \end{matrix} \middle| \frac{\left(16-x^2\right)^3}{108x^4}\right)$$

new
BSWZ

Some problems

- Given a linear differential equation **automatically** find its "hypergeometric-type" solutions.

  Promising work by Mark van Hoeij and his group
- What is the average distance travelled in five steps?

  $W_n(1) = n \int_0^\infty J_1(x) J_0(x)^{n-1} \frac{\mathrm{d}x}{x}$
- What more can be said about p_5?

  We know it satisfies a (non-modular) DE, as well as its expansion at zero.

  Conjecture: $p_5'''(0) = \frac{78}{225} p_5'(0) - \frac{12}{5\pi^4} \frac{1}{p_5'(0)}$
- Countless generalization ...

  higher dimensions, different step sizes, ...

THANK YOU!

- Slides for this talk will be available from my website:
  `http://arminstraub.com/talks`

J. Borwein, D. Nuyens, A. Straub, J. Wan
Some arithmetic properties of short random walk integrals
The Ramanujan Journal, Vol. 26, Nr. 1, 2011, p. 109-132

J. Borwein, A. Straub, J. Wan
Three-step and four-step random walk integrals
Experimental Mathematics — to appear

J. Borwein, A. Straub, J. Wan, W. Zudilin (appendix by D. Zagier)
Densities of short uniform random walks
Canadian Journal of Mathematics — to appear

...

Mahler measure

DEF (Logarithmic) Mahler measure of $p(x_1, \ldots, x_n)$:

$$\mu(p) := \int_0^1 \cdots \int_0^1 \log \left| p\left(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n}\right) \right| \mathrm{d}t_1 \mathrm{d}t_2 \ldots \mathrm{d}t_n$$

Mahler measure

DEF (Logarithmic) Mahler measure of $p(x_1, \ldots, x_n)$:

$$\mu(p) := \int_0^1 \cdots \int_0^1 \log \left| p\left(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n}\right) \right| \mathrm{d}t_1 \mathrm{d}t_2 \ldots \mathrm{d}t_n$$

LEM **Jensen**

$$\int_0^1 \log \left| \alpha + e^{2\pi i\, t} \right| \, \mathrm{d}t = \log\left(\max\{|\alpha|, 1\}\right)$$

Mahler measure

DEF (Logarithmic) Mahler measure of $p(x_1, \ldots, x_n)$:

$$\mu(p) := \int_0^1 \cdots \int_0^1 \log \left| p\left(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n}\right) \right| \mathrm{d}t_1 \mathrm{d}t_2 \ldots \mathrm{d}t_n$$

LEM Jensen

$$\int_0^1 \log \left| \alpha + e^{2\pi i\, t} \right| \, \mathrm{d}t = \log\left(\max\{|\alpha|, 1\}\right)$$

CONJ Lehmer, 1931 If $p(x)$ is not a product of cyclotomics then

$$\mu(p) \geqslant \mu(1 - x + x^3 - x^4 + x^5 - x^6 + x^7 - x^9 + x^{10}) = 0.162358.$$

Mahler measure

DEF (Logarithmic) Mahler measure of $p(x_1, \ldots, x_n)$:

$$\mu(p) := \int_0^1 \cdots \int_0^1 \log \left| p\left(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n}\right) \right| \mathrm{d}t_1 \mathrm{d}t_2 \ldots \mathrm{d}t_n$$

LEM Jensen

$$\int_0^1 \log \left| \alpha + e^{2\pi i\, t} \right| \, \mathrm{d}t = \log\left(\max\{|\alpha|, 1\}\right)$$

CONJ Lehmer, 1931 If $p(x)$ is not a product of cyclotomics then

$$\mu(p) \geqslant \mu(1 - x + x^3 - x^4 + x^5 - x^6 + x^7 - x^9 + x^{10}) = 0.162358.$$

EG Smyth, 1981

$$\mu(1 + x + y) = \frac{1}{\pi} \mathrm{Cl}_2\left(\frac{\pi}{3}\right)$$

$$\mu(1 + x + y + z) = \frac{7}{2} \frac{\zeta(3)}{\pi^2}$$

Mahler measure

DEF (Logarithmic) Mahler measure of $p(x_1, \ldots, x_n)$:

$$\mu(p) := \int_0^1 \cdots \int_0^1 \log \left| p\left(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n}\right) \right| \mathrm{d}t_1 \mathrm{d}t_2 \ldots \mathrm{d}t_n$$

LEM Jensen

$$\int_0^1 \log \left|\alpha + e^{2\pi i\, t}\right| \, \mathrm{d}t = \log\left(\max\{|\alpha|, 1\}\right)$$

CONJ Lehmer, 1931 If $p(x)$ is not a product of cyclotomics then $\mu(p) \geqslant \mu(1-x+x^3-x^4+x^5-x^6+x^7-x^9+x^{10}) = 0.162358.$

EG Smyth, 1981

$$\mu(1+x+y) = \frac{1}{\pi}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right) = W_3'(0)$$

$$\mu(1+x+y+z) = \frac{7}{2}\frac{\zeta(3)}{\pi^2} = W_4'(0)$$

Mahler measure and random walks

- $W_n(s) = \int_{[0,1]^n} \left| e^{2\pi x_1 i} + \ldots + e^{2\pi x_n i} \right|^s \mathrm{d}\boldsymbol{x}$
- $W_n'(0) = \mu(x_1 + \ldots + x_n) = \mu(1 + x_1 + \ldots + x_{n-1})$

Mahler measure and random walks

- $W_n(s) = \int_{[0,1]^n} \left| e^{2\pi x_1 i} + \ldots + e^{2\pi x_n i} \right|^s \mathrm{d}\boldsymbol{x}$
- $W_n'(0) = \mu(x_1 + \ldots + x_n) = \mu(1 + x_1 + \ldots + x_{n-1})$

EG
Rogers–Zudilin, 2011

Typical conjecture (Deninger, 1997):

$$\mu(1 + x + y + 1/x + 1/y) = \frac{15}{4\pi^2} L_E(2)$$

where L_E is the L-series for an elliptic curve of conductor 15.

Mahler measure and random walks

- $W_n(s) = \int_{[0,1]^n} \left| e^{2\pi x_1 i} + \ldots + e^{2\pi x_n i} \right|^s \mathrm{d}\boldsymbol{x}$
- $W_n'(0) = \mu(x_1 + \ldots + x_n) = \mu(1 + x_1 + \ldots + x_{n-1})$

EG Rogers–Zudilin, 2011

Typical conjecture (Deninger, 1997):

$$\mu(1 + x + y + 1/x + 1/y) = \frac{15}{4\pi^2} L_E(2)$$

where L_E is the L-series for an elliptic curve of conductor 15.

CONJ Rodriguez-Villegas

$$W_5'(0) \overset{?}{=} \left(\frac{15}{4\pi^2}\right)^{5/2} \int_0^\infty \left[\eta^3(e^{-3t})\eta^3(e^{-5t}) + \eta^3(e^{-t})\eta^3(e^{-15t})\right] t^3 \, \mathrm{d}t$$

CONJ Rodriguez-Villegas

$$W_6'(0) \overset{?}{=} \left(\frac{3}{\pi^2}\right)^3 \int_0^\infty \eta^2(e^{-t})\eta^2(e^{-2t})\eta^2(e^{-3t})\eta^2(e^{-6t}) \, t^4 \, \mathrm{d}t$$

- Representations for $W_n(s)$ give us, for instance,

$$\begin{aligned} W_n'(0) &= \log(2) - \gamma - \int_0^1 \left(J_0^n(x) - 1\right) \frac{\mathrm{d}x}{x} - \int_1^\infty J_0^n(x) \frac{\mathrm{d}x}{x} \\ &= \log(2) - \gamma - n \int_0^\infty \log(x) J_0^{n-1}(x) J_1(x) \mathrm{d}x. \end{aligned}$$

(Multiple) Mahler measure

DEF Multiple Mahler measure of polynomials $p_i(x_1,\ldots,x_n)$:

$$\mu(p_1,\ldots,p_k) := \int_{[0,1]^n} \prod_{i=1}^{k} \log\left|p_i\left(e^{2\pi i t_1},\ldots,e^{2\pi i t_n}\right)\right| \mathrm{d}\mathbf{t}$$

$$\mu_k(p) := \int_{[0,1]^n} \log^k \left|p\left(e^{2\pi i t_1},\ldots,e^{2\pi i t_n}\right)\right| \mathrm{d}\mathbf{t}$$

(Multiple) Mahler measure

DEF Multiple Mahler measure of polynomials $p_i(x_1, \ldots, x_n)$:

$$\mu(p_1, \ldots, p_k) := \int_{[0,1]^n} \prod_{i=1}^{k} \log \left| p_i \left(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n} \right) \right| \mathrm{d}\mathbf{t}$$

$$\mu_k(p) := \int_{[0,1]^n} \log^k \left| p \left(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n} \right) \right| \mathrm{d}\mathbf{t}$$

EG

$$W_n^{(k)}(0) = \mu_k(1 + x_1 + \ldots + x_{n-1})$$

Moments of a 3-step random walk

EG Borwein-Borwein-S-Wan

$$\mu_1(1+x+y) = \frac{3}{2\pi}\operatorname{Ls}_2\left(\frac{2\pi}{3}\right)$$

$$\mu_2(1+x+y) = \frac{3}{\pi}\operatorname{Ls}_3\left(\frac{2\pi}{3}\right) + \frac{\pi^2}{4}$$

$$\begin{aligned}\mu_3(1+x+y) &\overset{?}{=} \frac{6}{\pi}\operatorname{Ls}_4\left(\frac{2\pi}{3}\right) - \frac{9}{\pi}\operatorname{Cl}_4\left(\frac{\pi}{3}\right)\\ &\quad - \frac{\pi}{4}\operatorname{Cl}_2\left(\frac{\pi}{3}\right) - \frac{13}{2}\zeta(3)\end{aligned}$$

$$\begin{aligned}\mu_4(1+x+y) &\overset{?}{=} \frac{12}{\pi}\operatorname{Ls}_5\left(\frac{2\pi}{3}\right) - \frac{49}{3\pi}\operatorname{Ls}_5\left(\frac{\pi}{3}\right) + \frac{81}{\pi}\operatorname{Gl}_{4,1}\left(\frac{2\pi}{3}\right)\\ &\quad + 3\pi\operatorname{Gl}_{2,1}\left(\frac{2\pi}{3}\right) + \frac{2}{\pi}\zeta(3)\operatorname{Cl}_2\left(\frac{\pi}{3}\right)\\ &\quad + \operatorname{Cl}_2\left(\frac{\pi}{3}\right)^2 - \frac{29}{90}\pi^4\end{aligned}$$

Derivatives of moments

- Using the residues $r_{5,k} = \mathrm{Res}_{-2k-2}\, W_5$:

$$p_5(x) = \sum_{k=0}^{\infty} r_{5,k}\, x^{2k+1}$$

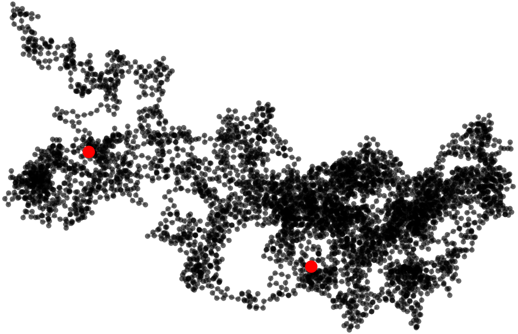
EG

$$r_{5,0} = \frac{16 + 1140 W_5'(0) - 804 W_5'(2) + 64 W_5'(4)}{225},$$

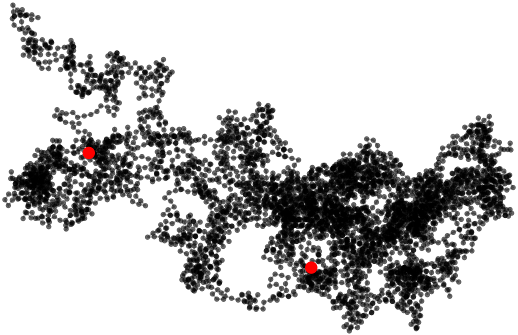
$$r_{5,1} = \frac{26 r_{5,0} - 16 - 20 W_5'(0) + 4 W_5'(2)}{225}.$$

- Unfortunately, the Mahler measure $W_5'(0)$ “cancels” out.

Drunken birds



Drunken birds



> *A drunk man will find his way home, but a drunk bird may get lost forever.*
>
> **Shizuo Kakutani**, *1911–2004*