Applications and evaluations of log-sine integrals

JonFest 2011

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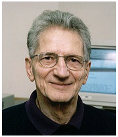
Based on joint work with Jon and, in parts, with:



David Borwein

James Wan

Wadim Zudilin

Definition

- (Generalized) **log-sine integrals**:

$$\mathrm{Ls}_n(\sigma) = -\int_0^\sigma \log^{n-1} \left|2 \sin \frac{\theta}{2}\right| \mathrm{d}\theta$$

$$\mathrm{Ls}_n^{(k)}(\sigma) = -\int_0^\sigma \theta^k \log^{n-1-k} \left|2 \sin \frac{\theta}{2}\right| \mathrm{d}\theta$$

L. Lewin
Polylogarithms and associated functions
North Holland, 1981

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$$\mathrm{Ls}_n^{(k)}(\sigma) = -\int_0^\sigma \theta^k \log^{n-1-k}\left|2\sin\frac{\theta}{2}\right| \mathrm{d}\theta$$

- Almost periodic:

$$\mathrm{Ls}_n^{(k)}(2m\pi) - \mathrm{Ls}_n^{(k)}(2m\pi - \sigma) = \sum_{j=0}^{k} (-1)^{k-j} (2m\pi)^j \binom{k}{j} \mathrm{Ls}_{n-j}^{(k-j)}(\sigma)$$

- So, without much loss, $0 \leqslant \sigma \leqslant \pi$

  Of special interest: $\pi/3$, $\pi/2$, $2\pi/3$, π

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Interesting and queer

It's not just us who find it intriguing that, for instance:

$$-\mathrm{Ls}_5(\pi) = \int_0^{\pi} \log^4 \left|2 \sin \tfrac{\theta}{2}\right| \, \mathrm{d}\theta = \frac{19}{240}\pi^5$$

$$-\mathrm{Ls}_4^{(1)}\left(\frac{\pi}{3}\right) = \int_0^{\pi/3} \theta \log^2 \left|2 \sin \tfrac{\theta}{2}\right| \, \mathrm{d}\theta = \frac{17}{6480}\pi^4$$

aspects of the trilogarithm in Chapter 6 are, I believe, also novel, as are the log-sine integrals of Chapter 6 and 7. These functions, too, have their queer special numerical results. Perhaps, to keep the torch burning, it would not be out of place to record, for example, the relation

$$\int_0^{\pi/3} \theta \log^2 (2 \sin \tfrac{1}{2}\theta)\, d\theta = 17\, \pi^4/6480;$$

maybe in its own way as interesting as any of Landen's curious formulas.

Basic log-sine integrals at π

An exponential generating function (Lewin)

$$-\frac{1}{\pi}\sum_{n=0}^{\infty}\mathrm{Ls}_{n+1}\left(\pi\right)\frac{\lambda^n}{n!}=\frac{\Gamma\left(1+\lambda\right)}{\Gamma^2\left(1+\frac{\lambda}{2}\right)}=\binom{\lambda}{\frac{\lambda}{2}}$$

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Example (Mathematica)

```
FullSimplify[D[-Pi Binomial[x,x/2], {x,5}] /.x->0]
```

$$\mathrm{Ls}_6(\pi) = \frac{45}{2}\pi\,\zeta(5) + \frac{5}{4}\pi^3\zeta(3)$$

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$$\mathrm{Ls}_6\left(\pi\right)=\frac{45}{2}\pi\,\zeta(5)+\frac{5}{4}\pi^3\zeta(3)$$

Example (Maple)

```
simplify(subs(x=0,diff(-Pi*binomial(x,x/2),x$5)))
```

Notation for polylogs

depth k
weight $w = a_1 + \ldots + a_k$

- Multiple polylogarithm:

$$\mathrm{Li}_{a_1,\ldots,a_k}(z) = \sum_{n_1 > \cdots > n_k > 0} \frac{z^{n_1}}{n_1^{a_1} \cdots n_k^{a_k}}$$

- Multiple zeta values:

$$\zeta(a_1, \ldots, a_k) = \mathrm{Li}_{a_1,\ldots,a_k}(1)$$

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- Multiple Clausen and Glaisher functions:

$$\mathrm{Cl}_{a_1,\ldots,a_k}(\theta) = \left\{ \begin{array}{ll} \operatorname{Im} \mathrm{Li}_{a_1,\ldots,a_k}(e^{i\theta}) & \text{if } w \text{ even} \\ \operatorname{Re} \mathrm{Li}_{a_1,\ldots,a_k}(e^{i\theta}) & \text{if } w \text{ odd} \end{array} \right\}$$

$$\mathrm{Gl}_{a_1,\ldots,a_k}(\theta) = \left\{ \begin{array}{ll} \operatorname{Re} \mathrm{Li}_{a_1,\ldots,a_k}(e^{i\theta}) & \text{if } w \text{ even} \\ \operatorname{Im} \mathrm{Li}_{a_1,\ldots,a_k}(e^{i\theta}) & \text{if } w \text{ odd} \end{array} \right\}$$

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Example

$$\mathrm{Cl}_2\left(\theta\right) = \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^2}$$

- M

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Notation for polylogs

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Example

$$\mathrm{Cl}_2\left(\theta\right) = \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^2}$$

$$\mathrm{Cl}_{4,1}\left(\theta\right) = \sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n^4} \sum_{k=1}^{n-1} \frac{1}{k}$$

- M
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$$\mathrm{Gl}_{a_1,\ldots,a_k}\left(\theta\right) = \left\{ \begin{array}{ll} \mathrm{Re}\,\mathrm{Li}_{a_1,\ldots,a_k}(e^{i\theta}) & \text{if } w \text{ even} \\ \mathrm{Im}\,\mathrm{Li}_{a_1,\ldots,a_k}(e^{i\theta}) & \text{if } w \text{ odd} \end{array} \right\}$$

A generating function

Theorem (Lewin, Borwein-S)

$$-\sum_{n,k\geqslant 0} \mathrm{Ls}_{n+k+1}^{(k)}(\pi)\,\frac{\lambda^n}{n!}\frac{(i\mu)^k}{k!} = i\sum_{n\geqslant 0}\binom{\lambda}{n}\frac{(-1)^n \mathrm{e}^{i\pi\frac{\lambda}{2}} - \mathrm{e}^{i\pi\mu}}{n+\mu-\frac{\lambda}{2}}$$

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Example

$$-\mathrm{Ls}_4^{(2)}(\pi) = \frac{\mathrm{d}^2}{\mathrm{d}\mu^2} \frac{\mathrm{d}}{\mathrm{d}\lambda} i \sum_{n\geqslant 0} \binom{\lambda}{n} \frac{(-1)^n \mathrm{e}^{i\pi\frac{\lambda}{2}} - \mathrm{e}^{i\pi\mu}}{\mu-\frac{\lambda}{2}+n} \Bigg|_{\lambda=\mu=0}$$

$$= 2\pi \sum_{n\geqslant 1} \frac{(-1)^{n+1}}{n^3} = \frac{3}{2}\pi\zeta(3)$$

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$$\frac{(-1)^\alpha}{\alpha!} \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^\alpha \binom{\lambda}{n}\bigg|_{\lambda=0} = \frac{(-1)^n}{n} \underbrace{\sum_{n>i_1>i_2>\ldots>i_{\alpha-1}} \frac{1}{i_1 i_2 \cdots i_{\alpha-1}}}_{H_{n-1}^{[\alpha-1]}}$$

Note: $\displaystyle\sum_{n\geqslant 1} \frac{(\pm 1)^n}{n^\beta} H_{n-1}^{[\alpha]} = \mathrm{Li}_{\beta,\{1\}^\alpha}(\pm 1)$

Thus $\mathrm{Ls}_n^{(k)}(\pi)$ evaluates in terms of the Nielsen polylogs $\mathrm{Li}_{\beta,\{1\}^\alpha}(\pm 1)$.

Log-sine integrals at general arguments

$$\mathrm{Li}_{3,1}(1) = \mathrm{Li}_{3,1}(\mathrm{e}^{i\tau}) - \int_1^{\mathrm{e}^{i\tau}} \frac{\mathrm{Li}_{2,1}(z)}{z}\,\mathrm{d}z$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\,\mathrm{Li}_{k,\ldots}(z) = \frac{1}{z}\,\mathrm{Li}_{k-1,\ldots}(z)$$

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Taking the real part:

$$\mathrm{Li}_{3,1}(1) = \mathrm{Gl}_{3,1}(\tau) + \tau\,\mathrm{Gl}_{2,1}(\tau) + \frac{1}{2}\mathrm{Ls}_4^{(1)}(\tau) + \frac{1}{32}\tau^4 - \frac{1}{12}\pi\tau^3 + \frac{1}{16}\pi^2\tau^2$$

Log-sine integrals at general arguments

Theorem (Borwein-S)

$$\zeta(k,\{1\}^n) - \sum_{j=0}^{k-2} \frac{(-i\tau)^j}{j!} \operatorname{Li}_{k-j,\{1\}^n}(e^{i\tau})$$
$$= \frac{(-i)^{k-1}}{(k-2)!} \frac{(-1)^n}{(n+1)!} \sum_{r,m} \binom{n+1}{r} \binom{r}{m} \left(\frac{i}{2}\right)^r (-\pi)^{r-m} \operatorname{Ls}_{n+k-(r-m)}^{(k+m-2)}(\tau)$$

Example ($n=2$, $k=3$)

$$6\zeta(3,1,1) - 6\operatorname{Cl}_{3,1,1}(\tau) - 6\tau \operatorname{Cl}_{2,1,1}(\tau)$$
$$= -\operatorname{Ls}_5^{(1)}(\tau) + \frac{3}{4}\operatorname{Ls}_5^{(3)}(\tau) - \frac{3}{2}\pi \operatorname{Ls}_4^{(2)}(\tau) + \frac{3}{4}\pi^2 \operatorname{Ls}_3^{(1)}(\tau)$$

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Thus $\operatorname{Ls}_n^{(k)}(\tau)$ evaluates in terms of Nielsen polylogs $\operatorname{Li}_{\beta,\{1\}^\alpha}(e^{i\tau})$.

Implementation

```
In[1]:= << "~/docs/math/mathematica/logsine.m"

LsToLi: evaluating log-sine integrals in polylogarithmic terms
     accompanying the paper "Special values of generalized log-sine integrals"
     -- Jonathan M. Borwein, University of Newcastle
     -- Armin Straub, Tulane University
     -- Version 1.3 (2011/03/11)
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In[2]:= Ls[4, 1, Pi / 3] // LsToLi
```

Out[2]= $-\frac{17\,\pi^4}{6480}$

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```
Ls[4, 1, Pi / 3] // LsToLi
```

Out[2]= $-\frac{17\,\pi^4}{6480}$

In[3]:=
```
$Assumptions = 0 < τ < Pi;
Ls[4, 1, τ] // LsToLi
```

Out[4]= $\frac{\pi^4}{180} - \frac{\pi^2\,\tau^2}{8} + \frac{\pi\,\tau^3}{6} - \frac{\tau^4}{16} - 2\,\tau\,\mathrm{Gl}[\{2, 1\}, \tau] - 2\,\mathrm{Gl}[\{3, 1\}, \tau]$

Automatic identity proving

Example (Zucker, 1985)

$$\mathrm{Ls}_6^{(3)}\left(\frac{\pi}{3}\right) - 2\,\mathrm{Ls}_6^{(1)}\left(\frac{\pi}{3}\right) = \frac{313}{204120}\pi^6$$

Numerical usage

```
ls52 = Ls[5, 2, 2 Pi / 3] // LsToLi
```

$$-\frac{8\,\pi^5}{1215} - \frac{8}{9}\,\pi^2\,\mathrm{Gl}\left[\{2,\,1\},\,\frac{2\,\pi}{3}\right] - \frac{8}{3}\,\pi\,\mathrm{Gl}\left[\{3,\,1\},\,\frac{2\,\pi}{3}\right] + 4\,\mathrm{Gl}\left[\{4,\,1\},\,\frac{2\,\pi}{3}\right]$$

```
N[ls52]
```

```
-0.518109
```

```
N[ls52, 200]
```

```
-0.518108786829680117347265638731696755021879668243153214067389472482464930592067915068 1
  7591796234263409228316887407062572713789701522832828830123805334443460155548241634968 7
  142642605456956152340876808 79
```

M. Kalmykov and A. Sheplyakov
lsjk - a C++ library for arbitrary-precision numeric evaluation of the generalized log-sine functions
Comput. Phys. Commun., 172(1):45–59, 2005

Numerical usage

```
N[ls52 /. Gl → GlN, 1500]

-0.518108786829680117347265638731696755021879668243153214067389472482464930592067915068 1\
 7591796234263409228316887407062572713789701522832828830123805334443460155548241634968 7\
 1426426054569561523408768087883301252744524532005650653916633546607642565939433325023 6\
 8704996964072618430077108019449120638389717243843114495652058348073506174420066399192 0\
 9369665418959139680545328024232441688700377283742072772714029093211428066255503314839 3\
 4346157017999680014851656153847980079446225103428769237020891516271578761307434220860 7\
 9568427226785119431613041566002524816863193071905942805038395349632095408925090936865 0\
 7634802118402367023958364480605724883286048232501257730562643059641954715004420327604 8\
 0010663468680842589408031170729595597489331212416860550548109691664311874770739977269 9\
 5685152764342004787909957595709564369273416476440874928297302997322623212162505556681 8\
 3011472959999435674673619440973336383204370234472148571069312485138137784268429835976 2\
 7902662692522141735091653899270134033573580034488504210827748452303627506494601119042 4\
 5981100513011527807257203511249099602357226702813690561684728688359473915676500087227 4\
 1171715732532324304909392481203408934403135504697147308473633542555204308358155331186 4\
 6643809518680164612459364379017456732824209636267198665451044904639579951516434428642 2\
 0863570350441280524570356600335366814158891280187202197951693897999883945925408859044 9\
 5838444532048484183137311323287438168230950278471856582525113193260785397060334327453 1\
 023502381511423660445880027683673379705
```

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Reductions of polylogarithms

- The (Nielsen) polylogarithms satisfy various relations:

Example (Weight 4, $2\sigma = \tau - \pi$)

$$\mathrm{Cl}_{3,1}(\tau) = \mathrm{Cl}_4(\tau) - \sigma\,\mathrm{Cl}_3(\tau) + \sigma\zeta(3)$$

$$\mathrm{Gl}_4(\tau) = -\frac{\sigma^4}{3} + \zeta(2)\sigma^2 - \frac{7\zeta(4)}{8}$$

$$\mathrm{Gl}_{2,1,1}(\tau) = \frac{1}{2}\,\mathrm{Gl}_{3,1}(\tau) + \sigma\,\mathrm{Gl}_{2,1}(\tau) + \frac{\sigma^4}{6} - \frac{\zeta(4)}{16}$$

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- For the special argument $\tau = \pi/3$ there are additional relations:

$$\mathrm{Cl}_{2,1,1}\left(\frac{\pi}{3}\right) = \mathrm{Cl}_4\left(\frac{\pi}{3}\right) - \frac{\pi^2}{18}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{\pi}{9}\zeta(3)$$

Log-sine integrals at $\pi/3$

- $\mathrm{Ls}_n^{(k)}\left(\frac{\pi}{3}\right)$ evaluates in terms of polylogarithms at $\omega = \mathrm{e}^{i\pi/3}$ — the sixth root of unity
- Crucial: $1 - \omega = \overline{\omega} = \omega^2$

$$\mathrm{Li}_{2,1,1}(x) = \frac{\pi^4}{90} - \frac{1}{6}\log(1-x)^3 \log x - \frac{1}{2}\log(1-x)^2 \,\mathrm{Li}_2(1-x) + \log(1-x)\,\mathrm{Li}_3(1-x) - \mathrm{Li}_4(1-x)$$

- Currently, our implementation has a table of extra reductions built in — a systematic study is under way

Mahler measure

- Mahler measure of $p(x_1, \ldots, x_n)$:

$$\mu(p) := \int_0^1 \cdots \int_0^1 \log \left| p\left(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n}\right) \right| \mathrm{d}t_1 \mathrm{d}t_2 \ldots \mathrm{d}t_n$$

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- Jensen's formula:

$$\int_0^1 \log \left| \alpha + e^{2\pi i\, t} \right| \, \mathrm{d}t = \log\left(\max\{|\alpha|, 1\}\right)$$

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Typical conjecture (Deninger, 1997)—proven by Rogers-Zudilin, 2011

$$\mu(1 + x + y + 1/x + 1/y) = \frac{15}{4\pi^2} L_E(2)$$

where L_E is the L-series for an elliptic curve of conductor 15.

Classical Mahler measures

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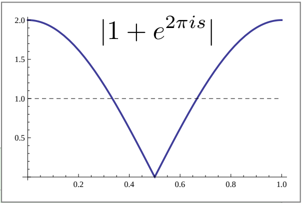

Example (Smyth's evaluations revisited)

$$\mu(1 + x + y) = \int_0^1 \int_0^1 \log\left|1 + e^{2\pi i s} + e^{2\pi i t}\right| \mathrm{d}t\mathrm{d}s$$

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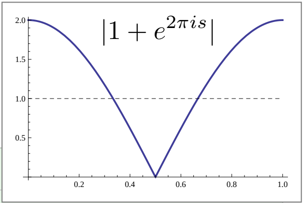

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$$\begin{aligned}\mu(1+x+y) &= \int_0^1\int_0^1 \log\left|1+e^{2\pi is}+e^{2\pi it}\right| \mathrm{d}t\mathrm{d}s \\ &= 2\int_0^{1/3} \log\left|1+e^{2\pi is}\right| \mathrm{d}s\end{aligned}$$

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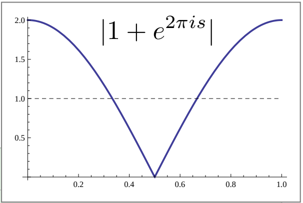

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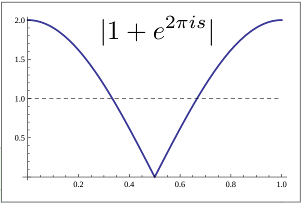

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$$\mu(1+x+y+z) = \ldots = -\frac{2}{\pi^2}\operatorname{Ls}_3^{(1)}(\pi) = \frac{7}{2}\frac{\zeta(3)}{\pi^2}$$

A multiple Mahler measure

$$\mu(p_1, \ldots, p_k) := \int_0^1 \cdots \int_0^1 \prod_{i=1}^{k} \log \left| p_i \left(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n} \right) \right| \mathrm{d}t_1 \mathrm{d}t_2 \ldots \mathrm{d}t_n$$

An example considered by Sasaki put in log-sine form

$$\begin{aligned} \mu_k(1+x+y_*) &:= \mu(1+x+y_1, 1+x+y_2, \ldots, 1+x+y_k) \\ &= \int_{1/6}^{5/6} \log^k \left| 1 - e^{2\pi i\, t} \right| \mathrm{d}t \\ &= \frac{1}{\pi} \mathrm{Ls}_{k+1}\left(\frac{\pi}{3}\right) - \frac{1}{\pi} \mathrm{Ls}_{k+1}(\pi) \end{aligned}$$

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Example (Kurokawa-Lalín-Ochiai)

$$\mu_2(1+x+y_*) = \frac{\pi^2}{54}$$

Inverse binomial sums

A very classical example

$$\sum_{j=1} \frac{1}{j^4} \frac{1}{\binom{2j}{j}} = \frac{17}{36} \zeta(4)$$

Inverse binomial sums

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This is "explained" by a general relation:

Theorem (Borwein-Broadhurst-Kamnitzer)

$$\sum_{j=1}^{\infty} \frac{1}{j^{n+2}} \frac{1}{\binom{2j}{j}} = -\frac{(-2)^n}{n!} \operatorname{Ls}_{n+2}^{(1)}\left(\frac{\pi}{3}\right)$$

J. M. Borwein, D. J. Broadhurst, and J. Kamnitzer.
Central binomial sums, multiple Clausen values, and zeta values.
Experimental Mathematics, 10(1):25–34, 2001.

Inverse binomial sums—next level

$u = 4\sin^2\frac{\theta}{2}$ (so $u = 1$ if $\theta = \frac{\pi}{3}$)
$l_\theta = \log\left(2\sin\frac{\theta}{2}\right)$, $L_\theta = \log\left(2\cos\frac{\theta}{2}\right)$

The connection is deeper:

Example (Davydychev-Kalmykov)

$$\sum_{j=1}^{\infty} \frac{u^j}{j^{n+2}} \frac{1}{\binom{2j}{j}} = -\sum_{m=0}^{n} \frac{(-2)^m (2l_\theta)^{n-m}}{m!(n-m)!} \mathrm{Ls}_{m+2}^{(1)}(\theta)$$

A. Davydychev and M. Kalmykov.
Massive Feynman diagrams and inverse binomial sums.
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$$\sum_{j=1}^{\infty} \frac{u^j}{j^{n+2}} \frac{1}{\binom{2j}{j}} S_2(j-1) = -\frac{1}{6} \sum_{m=0}^{n} \frac{(-2)^m (2l_\theta)^{n-m}}{m!(n-m)!} \mathrm{Ls}_{m+4}^{(3)}(\theta)$$

- Here: $S_a(j) = \sum_{i=1}^{j} \frac{1}{j^a}$

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$$\sum_{j=1}^{\infty} \frac{u^j}{j} \frac{1}{\binom{2j}{j}} S_1(j-1) S_1(2j-1) = \tan\frac{\theta}{2} \Big\{ 5\,\mathrm{Ls}_3(\pi-\theta) - 5\,\mathrm{Ls}_3(\pi) - \mathrm{Ls}_3(\theta) + \frac{1}{2}\mathrm{Ls}_3(2\theta) - 2\,\mathrm{Ls}_2(\theta)\,L_\theta + 2\,\mathrm{Ls}_2(\pi-\theta)\,l_\theta - 8\,\mathrm{Ls}_2(\pi-\theta)\,L_\theta - 2\theta l_\theta L_\theta + 4\theta L_\theta^2 + \frac{1}{12}\theta^3 \Big\}$$

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ε-expansion of a hypergeometric function

- $[\varepsilon^n]\ {}_3F_2\left(\begin{matrix}\frac{\varepsilon+2}{2}, \frac{\varepsilon+2}{2}, \frac{\varepsilon+2}{2} \\ 1, \frac{\varepsilon+3}{2}\end{matrix}\middle|\frac{1}{4}\right)$

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  where the inner sum is over all $m_1, \ldots, m_n \geqslant 0$ such that $m_1 + 2m_2 + \ldots + nm_n = n$, and $A_{k,j} = S_k(2j-1) - 2^{2-k} S_k(j-1) - 1$

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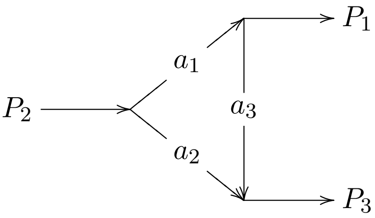
Example

$$[\varepsilon]\ {}_3F_2\left(\begin{matrix}\frac{\varepsilon+2}{2}, \frac{\varepsilon+2}{2}, \frac{\varepsilon+2}{2} \\ 1, \frac{\varepsilon+3}{2}\end{matrix}\middle|\frac{1}{4}\right) = -\sum_{j=1}^{\infty} \frac{2}{j} \frac{1}{\binom{2j}{j}} [S_1(2j-1) - 2S_1(j-1) - 1]$$
$$= \frac{2}{3\sqrt{3}} \left[\pi - \pi \log 3 + \mathrm{Ls}_2\left(\frac{\pi}{3}\right)\right]$$

Feynman diagrams

- 

Propagator associated to the index a_1 has mass m

$P_1^2 = P_3^2 = 0$ and
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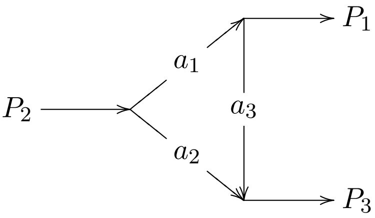
- 

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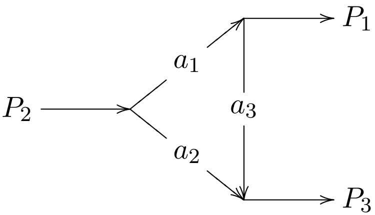
- 

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- In the special case $a_1 = a_2 = a_3 = 1$ and $D = 4 - 2\epsilon$ this becomes:

$$-(m^2)^{-1-\epsilon} \Gamma(\epsilon - 1) {}_2F_1 \left(\begin{matrix} 1+\epsilon, 1 \\ 2-\epsilon \end{matrix} \middle| \frac{s}{m^2} \right)$$

Moments of random walks

- Consider a random walk in the plane with n steps of length 1 chosen in uniformly random directions.
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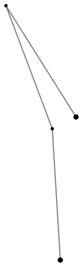

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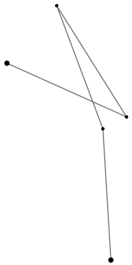

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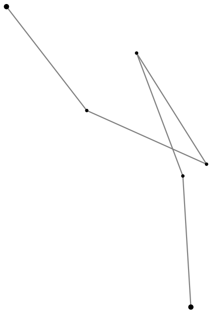

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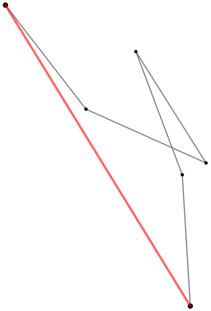

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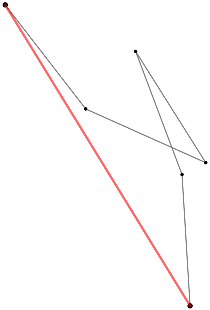

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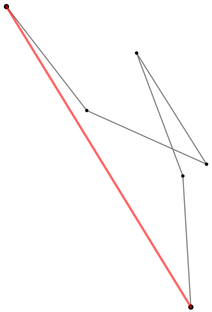
$$W_n(s) := \int_{[0,1]^n} \left| \sum_{k=1}^{n} e^{2\pi x_k i} \right|^s \mathrm{d}\boldsymbol{x}$$

In particular, $W_n(1)$ is the average distance after n steps.

Moments of random walks



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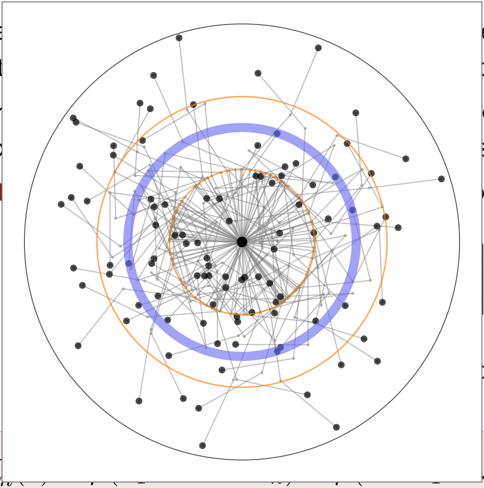
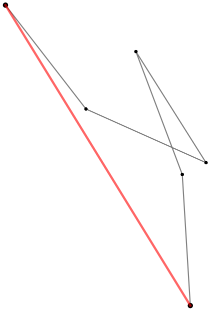
$$W_n'(0) = \mu(x_1 + \ldots + x_n) = \mu(1 + x_1 + \ldots + x_{n-1})$$

Moments of random walks





- Conside … eps of lengt … ons.
- We wer … es and mo … steps.
- The s-t … s is:

$$\ldots \Big|^{s} \, \mathrm{d}\boldsymbol{x}$$

In parti … ter n steps.

$$W_n \ldots + x_{n-1})$$

Moments of a 3-step random walk

Theorem (Borwein-S-Wan-Zudilin, 2010)

$$W_3(s) = \frac{\sqrt{3}}{2\pi}\, 3^{s+1}\, \frac{\Gamma(1+\frac{s}{2})^2}{\Gamma(s+2)}\, {}_3F_2\left(\left.\begin{matrix}\frac{s+2}{2}, \frac{s+2}{2}, \frac{s+2}{2} \\ 1, \frac{s+3}{2}\end{matrix}\right| \frac{1}{4}\right)$$

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- $W_3^{(k)}(0) = \mu_k(1+x+y)$ is the **higher Mahler measure**:

$$\mu_k(1+x+y) = \mu(\underbrace{1+x+y, 1+x+y, \ldots}_{k \text{ many}})$$

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Moments of a 3-step random walk

Example (Borwein-Borwein-S-Wan)

$$\mu_1(1+x+y) = \frac{3}{2\pi}\,\mathrm{Ls}_2\left(\frac{2\pi}{3}\right)$$

$$\mu_2(1+x+y) = \frac{3}{\pi}\,\mathrm{Ls}_3\left(\frac{2\pi}{3}\right) + \frac{\pi^2}{4}$$

$$\mu_3(1+x+y) \stackrel{?}{=} \frac{6}{\pi}\,\mathrm{Ls}_4\left(\frac{2\pi}{3}\right) - \frac{9}{\pi}\,\mathrm{Cl}_4\left(\frac{\pi}{3}\right) - \frac{\pi}{4}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{13}{2}\zeta(3)$$

$$\mu_4(1+x+y) \stackrel{?}{=} \frac{12}{\pi}\,\mathrm{Ls}_5\left(\frac{2\pi}{3}\right) - \frac{49}{3\pi}\,\mathrm{Ls}_5\left(\frac{\pi}{3}\right) + \frac{81}{\pi}\,\mathrm{Gl}_{4,1}\left(\frac{2\pi}{3}\right) + 3\pi\,\mathrm{Gl}_{2,1}\left(\frac{2\pi}{3}\right) + \frac{2}{\pi}\zeta(3)\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \mathrm{Cl}_2\left(\frac{\pi}{3}\right)^2 - \frac{29}{90}\pi^4$$

THANK YOU!

- Slides for this talk will be available from my website:
  `http://arminstraub.com/talks`

J. M. Borwein and A. Straub.
Special values of generalized log-sine integrals.
Proceedings of ISSAC 2011 (International Symposium on Symbolic and Algebraic Computation), 2011.
arXiv:1103.4298.

J. M. Borwein and A. Straub.
Log-sine evaluations of Mahler measures.
J. Aust Math. Soc., 2011.
arXiv:1103.3893.

D. Borwein, J. M. Borwein, A. Straub, and J. Wan.
Log-sine evaluations of Mahler measures, II.
Preprint, 2011.
arXiv:1103.3035.

...

A comment on notation

First, we didn't like the extra sign in $\mathrm{Ls}_n(\sigma) = -\int_0^\sigma \log^{n-1}\left|2\sin\frac{\theta}{2}\right| \mathrm{d}\theta \ldots$

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itself readily to such extension. The usual trap for a writer confronted with as confused a situation as this is for him to throw up his hands in despair and then proceed to add to the muddle by inventing his own notation. I have leant over

Happy birthday, Jon!

and thank you for 25 months (and counting)
of very enjoyable math together