Hypergeometric evaluations of the densities of short random walks

SIAM Conference on Applied Algebraic Geometry

North Carolina State University

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October 6, 2011

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Random walks in the plane

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- We are interested in the distance traveled in n steps.

> Denote the probability **density** of this distance by $p_n(x)$.

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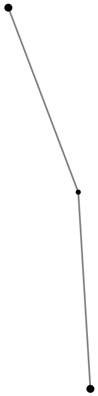

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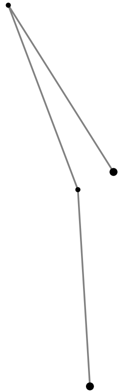

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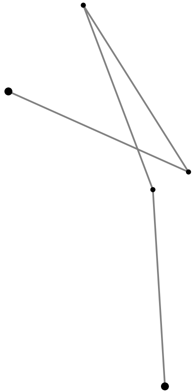

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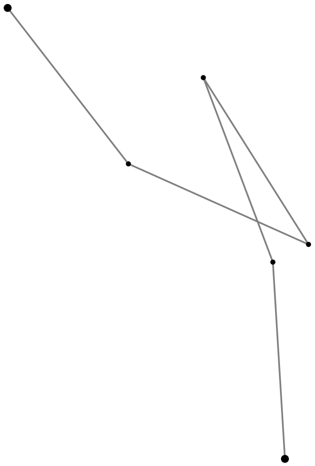

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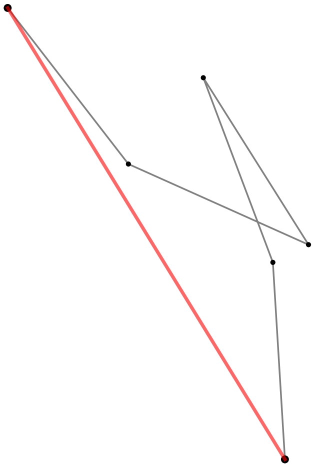

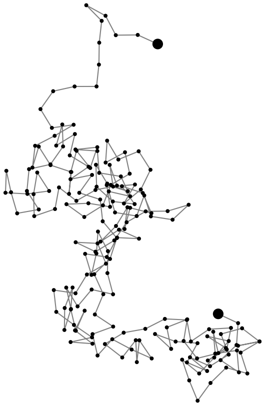

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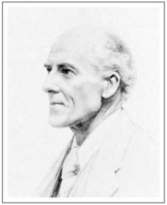
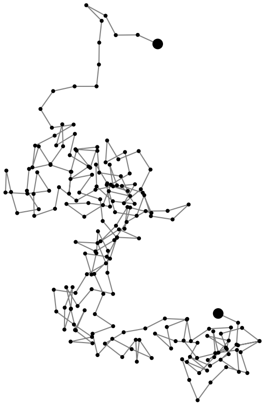
  This famous question coined the term **random walk**.

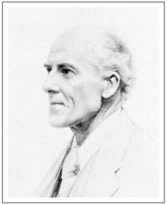


History and long walks

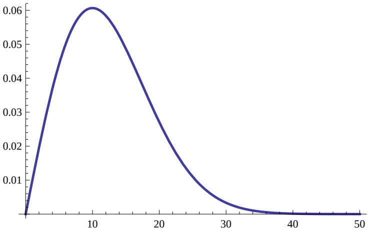


- Karl Pearson asked for $p_n(x)$ in Nature in 1905. This famous question coined the term **random walk**.
- Asymptotic answer by Lord Rayleigh:

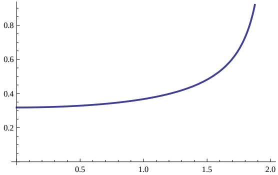
$$p_n(x) \approx \frac{2x}{n} e^{-x^2/n}$$



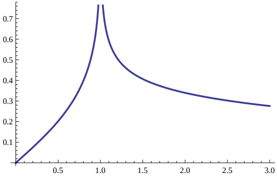
- For instance, $p_{200}(x)$:



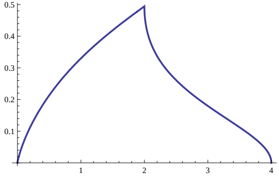
p_2



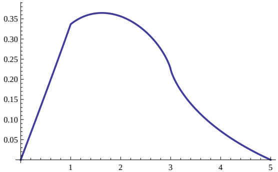
p_3



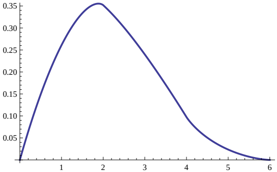
p_4



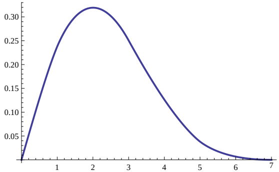
p_5



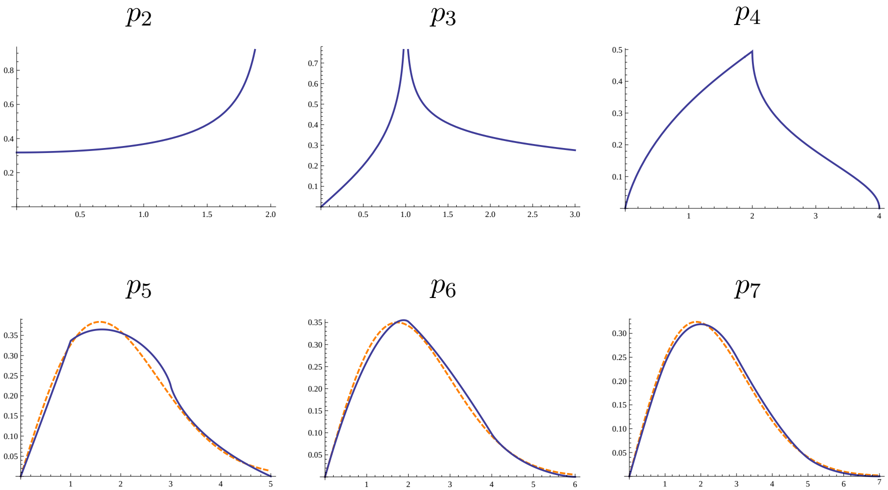
p_6



p_7



Densities of short walks



Classical results on the densities

$$p_2(x) = \frac{2}{\pi\sqrt{4-x^2}}$$ easy

$$p_3(x) = \operatorname{Re}\left(\frac{\sqrt{x}}{\pi^2}\,K\left(\sqrt{\frac{(x+1)^3(3-x)}{16x}}\right)\right)$$ G. J. Bennett 1905

$$\vdots$$

$$p_n(x) = \int_0^\infty xtJ_0(xt)J_0^n(t)\,\mathrm{d}t$$ J. C. Kluyver 1906

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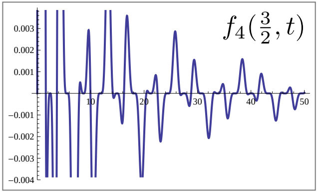
Experimentally, we observed that $p_4(x)$ satisfies an ODE.

The Bessel integral and some difficulties

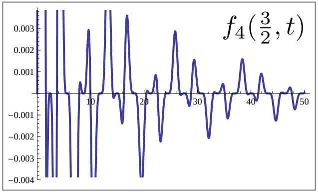
- $p_4(x) = \int_0^\infty \underbrace{xtJ_0(xt)J_0^4(t)}_{=:f_4(x,t)}\,\mathrm{d}t$



The Bessel integral and some difficulties

- $p_4(x) = \int_0^\infty \underbrace{xtJ_0(xt)J_0^4(t)}_{=:f_4(x,t)}\,\mathrm{d}t$
- **Creative telescoping** finds A, B so that

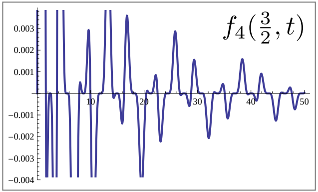
$$\left(A + \frac{\mathrm{d}}{\mathrm{d}t}\cdot B\right)\cdot f_4(x,t) = 0$$



$$A = (x-4)(x-2)x^3(x+2)(x+4)\frac{\mathrm{d}^3}{\mathrm{d}x^3} + 6x^4\left(x^2-10\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x\left(7x^4-32x^2+64\right)\frac{\mathrm{d}}{\mathrm{d}x} + \left(x^2-8\right)\left(x^2+8\right)$$

$$B = x^2t^3\frac{\mathrm{d}^4}{\mathrm{d}t^4} - 5x^3t^2\frac{\mathrm{d}^3}{\mathrm{d}t^3}\frac{\mathrm{d}}{\mathrm{d}x} + 7x^2t^2\frac{\mathrm{d}^3}{\mathrm{d}t^3} - x^2t(10x^2t^2-20t^2-1)\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 5x^3(2x^2t^2-12t^2-1)\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}x} - 4x^2(5x^2t^2-15t^2-1)\frac{\mathrm{d}}{\mathrm{d}t} - 5x^3(2x^2t^2-12t^2-1)/t\frac{\mathrm{d}}{\mathrm{d}x} + x^2(5x^4t^4-60x^2t^4+64t^4-28t^2-4)/t.$$

The Bessel integral and some difficulties



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- Switching orders:

$$\begin{aligned} A \cdot \int_0^T f_4(x,t)\mathrm{d}t &= \int_0^T A \cdot f_4(x,t)\mathrm{d}t \\ &= -B \cdot f_4(x,T) \end{aligned}$$

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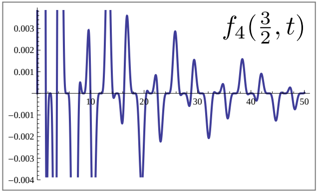
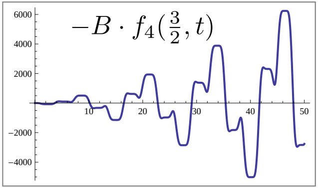
- But for $T=\infty$ the order can't be changed, and the RHS does not converge





Moments of random walks

- sth moment $W_n(s)$ of the density p_n:

$$W_n(s) = \int_0^\infty x^s p_n(x)\,\mathrm{d}x = \int_{[0,1]^n} \left| \sum_{k=1}^n e^{2\pi x_k i} \right|^s \mathrm{d}\mathbf{x}$$

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Combinatorial evaluation (Borwein-Nuyens-S-Wan, 2010)

$$W_n(2k) = \sum_{a_1+\cdots+a_n=k} \binom{k}{a_1,\ldots,a_n}^2$$

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- Inevitable **recursions**

$K \cdot f(k) = f(k+1)$

$$\left[(k+2)^2 K^2 - (10k^2+30k+23)K + 9(k+1)^2\right] \cdot W_3(2k) = 0$$
$$\left[(k+2)^3 K^2 - (2k+3)(10k^2+30k+24)K + 64(k+1)^3\right] \cdot W_4(2k) = 0$$

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- Via **Carlson's Theorem** these become functional equations

Crashcourse on the Mellin transform

- Mellin transform $F(s)$ of $f(x)$:
  $\mathcal{M}\left[f; s\right] = \int_0^\infty x^{s-1} f(x) \, \mathrm{d}x$

$$\boxed{W_n(s-1) = \mathcal{M}\left[p_n; s\right]}$$

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- Functional properties:
  - $\mathcal{M}\left[x^\mu f(x); s\right] = F(s+\mu)$
  - $\mathcal{M}\left[D_x f(x); s\right] = -(s-1)F(s-1)$
  - $\mathcal{M}\left[-\theta_x f(x); s\right] = sF(s)$

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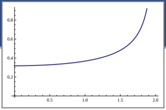
- Poles of $F(s)$ left of strip $\implies$ asymptotics of $f(x)$ at zero

  $\frac{1}{(s+m)^{n+1}}$ $\qquad$ $\frac{(-1)^n}{n!} x^m (\log x)^n$

Mellin approach illustrated for p_2



- $W_2(2k) = \binom{2k}{k}$

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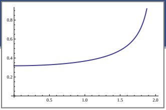

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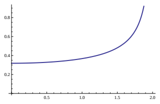

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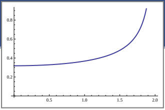

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- Hence: $p_2(x) = \dfrac{C}{\sqrt{4 - x^2}}$

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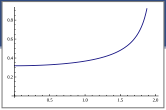

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$$\begin{aligned} W_2(s) &= \frac{1}{\pi}\frac{1}{s+1} + O(1) \text{ as } s \to -1 \\ p_2(x) &= \frac{1}{\pi} + O(x) \text{ as } x \to 0^+ \end{aligned}$$

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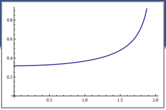

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- Taken together: $p_2(x) = \dfrac{2}{\pi\sqrt{4-x^2}}$

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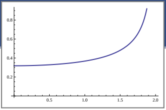

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$W_2(s)$ has poles at $s = -2k-1$ with residue $\dfrac{1}{\pi 2^{4k}}\dbinom{2k}{k}$

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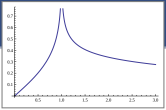
p_3 in hypergeometric form



- $W_3(s)$ has simple poles at $-2k-2$ with residue

$$\frac{2}{\pi\sqrt{3}}\,\frac{W_3(2k)}{3^{2k}}$$

$$p_3(x) = \frac{2x}{\pi\sqrt{3}} \sum_{k=0}^{\infty} W_3(2k) \left(\frac{x}{3}\right)^{2k} \qquad \text{for } 0 \leqslant x \leqslant 1$$

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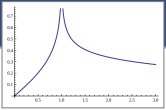

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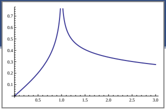
- $W_3(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j}$ is an **Apéry-like** sequence

p_3 in hypergeometric form



- $W_3(s)$ has simple poles at $-2k-2$ with residue

$$\frac{2}{\pi\sqrt{3}} \frac{W_3(2k)}{3^{2k}}$$

$$p_3(x) = \frac{2x}{\pi\sqrt{3}} \sum_{k=0}^{\infty} W_3(2k) \left(\frac{x}{3}\right)^{2k} \qquad \text{for } 0 \leqslant x \leqslant 1$$

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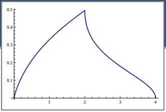
$$p_3(x) = \frac{2\sqrt{3}x}{\pi\left(3+x^2\right)} \, {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \frac{x^2\left(9-x^2\right)^2}{\left(3+x^2\right)^3}\right)$$

- Easy to verify once found
- Holds for $0 \leqslant x \leqslant 3$

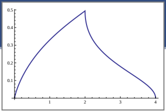
p_4 and its asymptotics at zero



- $W_4(s)$ has double poles at $-2k-2$:

$$W_4(s) = \frac{s_{4,k}}{(s+2k+2)^2} + \frac{r_{4,k}}{s+2k+2} + O(1) \quad \text{as } s \to -2k-2$$

p_4 and its asymptotics at zero



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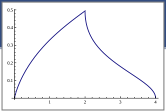
p_4 and its asymptotics at zero



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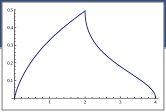
  $r_{4,k}$ known recursively

$$W_4(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} \binom{2n-2j}{n-j}$$

Domb numbers

p_4 and its asymptotics at zero



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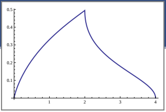
Domb numbers

Generating function for Domb numbers (Chan-Chan-Liu, 2004; Rogers, 2009)

$$\sum_{k=0}^{\infty} W_4(2k) z^k = \frac{1}{1-4z}\, {}_3F_2\left(\left.\begin{matrix}\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\\ 1, 1\end{matrix}\,\right|\, \frac{108z^2}{(1-4z)^3}\right)$$



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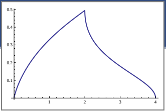
$r_{4,k}$ known recursively

Domb numbers

$$W_4(s) = \frac{3}{2\pi^2}\frac{1}{(s+2)^2} + \frac{9\log 2}{2\pi^2}\frac{1}{s+2} + O(1) \quad \text{as } s \to -2$$

$$p_4(x) = -\frac{3}{2\pi^2} x\log(x) + \frac{9\log 2}{2\pi^2} x + O(x^3) \quad \text{as } x \to 0^+$$

p_4 and its differential equation

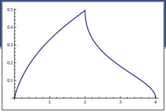


$$\left[(s+4)^3 S^4 - 4(s+3)(5s^2+30s+48)S^2 + 64(s+2)^3\right] \cdot W_4(s) = 0$$

translates into $A_4 \cdot p_4(x) = 0$ where A_4 is

$$A_4 = x^4(\theta+1)^3 - 4x^2\theta(5\theta^2+3) + 64(\theta-1)^3$$

p_4 and its differential equation



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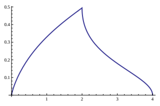
$$\begin{aligned} A_4 &= x^4(\theta+1)^3 - 4x^2\theta(5\theta^2+3) + 64(\theta-1)^3 \\ &= (x-4)(x-2)x^3(x+2)(x+4)D_x^3 + 6x^4\left(x^2-10\right)D_x^2 \\ &\quad + x\left(7x^4 - 32x^2 + 64\right)D_x + \left(x^2-8\right)\left(x^2+8\right) \end{aligned}$$



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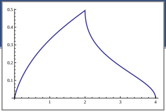
Care needed!

$p_4(x) \approx C\sqrt{4-x}$ as $x \to 4^-$. Thus p_4'' is not locally integrable and does not have a Mellin transform in the classical sense.

p_4 in hypergeometric form



Theorem (Borwein-S-Wan-Zudilin, 2011)

For $2 \leqslant x \leqslant 4$,
$$p_4(x) = \frac{2}{\pi^2} \frac{\sqrt{16-x^2}}{x} \, {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{5}{6}, \frac{7}{6} \end{matrix} \middle| \frac{\left(16-x^2\right)^3}{108x^4} \right).$$

- Again, easily (if tediously) provable once found

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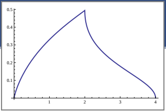

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- Again, easily (if tediously) provable once found
- Quite marvelously, as first observed numerically:

Theorem (Borwein-S-Wan-Zudilin, 2011)

For $0 \leqslant x \leqslant 4$,
$$p_4(x) = \frac{2}{\pi^2} \frac{\sqrt{16-x^2}}{x} \, \mathrm{Re} \, {}_3F_2\left(\left. \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{5}{6}, \frac{7}{6} \end{matrix} \right| \frac{\left(16-x^2\right)^3}{108x^4} \right).$$

p_4 in hypergeometric form — motivation

- $y_0(z) = \dfrac{1}{1-4z}\, {}_3F_2\left(\begin{matrix}\frac{1}{3}, \frac{1}{2}, \frac{2}{3} \\ 1, 1\end{matrix}\middle| \dfrac{108z^2}{(1-4z)^3}\right)$ is the analytic solution of

$$\left[64z^2(\theta+1)^3 - 2z(2\theta+1)(5\theta^2+5\theta+2) + \theta^3\right] \cdot y(z) = 0 \qquad \text{(DE)}$$

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$$p_4(x) = -\frac{3x}{4\pi^2}\, y_1\left(\frac{x^2}{64}\right)$$

where $y_1(z)$ solves (DE) and $y_1(z) - y_0(z)\log(z) \in z\mathbb{Q}[[z]]$

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p_4 in hypergeometric form — motivation

- $y_0(z) = \dfrac{1}{1-4z}\, {}_3F_2\left(\begin{matrix}\frac13, \frac12, \frac23 \\ 1, 1\end{matrix}\,\middle|\, \dfrac{108z^2}{(1-4z)^3}\right)$ is the analytic solution of

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- As $x \to 4$ then $z = \frac{x^2}{64} \to \frac14$ and $t = \frac{108z^2}{(1-4z)^3} \to \infty$
- Basis at ∞ for the hypergeometric equation of ${}_3F_2\left(\begin{matrix}\frac13, \frac12, \frac23 \\ 1, 1\end{matrix}\,\middle|\, t\right)$:

$$t^{-1/3}{}_3F_2\left(\begin{matrix}\frac13, \frac13, \frac13 \\ \frac23, \frac56\end{matrix}\,\middle|\, \frac1t\right), \quad t^{-1/2}{}_3F_2\left(\begin{matrix}\frac12, \frac12, \frac12 \\ \frac56, \frac76\end{matrix}\,\middle|\, \frac1t\right), \quad t^{-2/3}{}_3F_2\left(\begin{matrix}\frac23, \frac23, \frac23 \\ \frac43, \frac76\end{matrix}\,\middle|\, \frac1t\right)$$

p_4 and its modularity

Theorem (Chan-Zudilin, 2010)

$$y_0\left(-\frac{\eta(2\tau)^6\eta(6\tau)^6}{\eta(\tau)^6\eta(3\tau)^6}\right)=\frac{\eta(\tau)^4\eta(3\tau)^4}{\eta(2\tau)^2\eta(6\tau)^2}$$

- η is the **Dedekind eta function**: $q=e^{2\pi i\tau}$

$$\eta(\tau)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n)=q^{1/24}\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n+1)/2}$$

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- For $\tau=-1/2+iy$ and $y>0$:

$$p_4\left(8i\frac{\eta(2\tau)^3\eta(6\tau)^3}{\eta(\tau)^3\eta(3\tau)^3}\right)=\frac{6(2\tau+1)}{\pi}\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)$$

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If $f(\tau)$ is a modular form and $g(\tau)$ a modular function.
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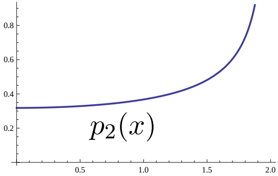
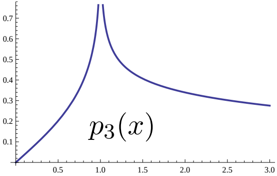
- For $\tau=-1/2+iy$ and $y>0$:

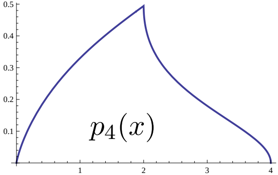
$\tau=\frac{\sqrt{-5/3}-1}{2}$ gives $p_4(1)$

$$p_4\left(8i\frac{\eta(2\tau)^3\eta(6\tau)^3}{\eta(\tau)^3\eta(3\tau)^3}\right)=\frac{6(2\tau+1)}{\pi}\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)$$

Hypergeometric formulae summarized







$$p_2(x) = \frac{2}{\pi\sqrt{4-x^2}}$$ easy

$$p_3(x) = \frac{2\sqrt{3}}{\pi}\frac{x}{(3+x^2)}\,{}_2F_1\left(\left.\begin{matrix}\frac{1}{3},\frac{2}{3}\\ 1\end{matrix}\right|\frac{x^2\left(9-x^2\right)^2}{\left(3+x^2\right)^3}\right)$$ classical with a spin

$$p_4(x) = \frac{2}{\pi^2}\frac{\sqrt{16-x^2}}{x}\,\mathrm{Re}\,{}_3F_2\left(\left.\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{5}{6},\frac{7}{6}\end{matrix}\right|\frac{\left(16-x^2\right)^3}{108x^4}\right)$$ new BSWZ

General problem

Research problem

Given a linear differential equation **automatically** find its "hypergeometric-type" solutions.

- Promising work by Mark van Hoeij and his group

Densities in general

Theorem (Borwein-S-Wan-Zudilin, 2011)

- *The density* p_n *satisfies a DE of order* $n-1$.
- p_n *is real analytic except at* 0 *and the integers* $n, n-2, n-4, \ldots$.

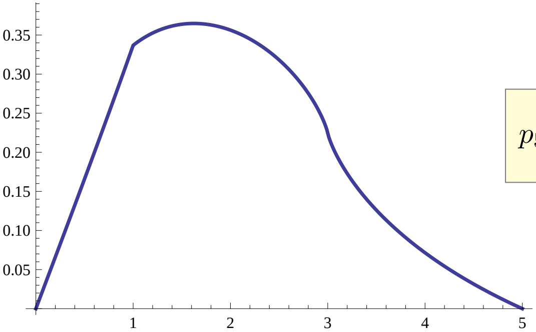
The second statement relies on an explicit recursion by Verrill (2004) as well as the combinatorial identity

$$\sum_{\substack{0 \leqslant m_1, \ldots, m_j < n/2 \\ m_i < m_{i+1}}} \prod_{i=1}^{j} (n - 2m_i)^2 = \sum_{\substack{1 \leqslant \alpha_1, \ldots, \alpha_j \leqslant n \\ \alpha_i \leqslant \alpha_{i+1} - 2}} \prod_{i=1}^{j} \alpha_i (n + 1 - \alpha_i).$$

First proven by Djakov-Mityagin (2004).

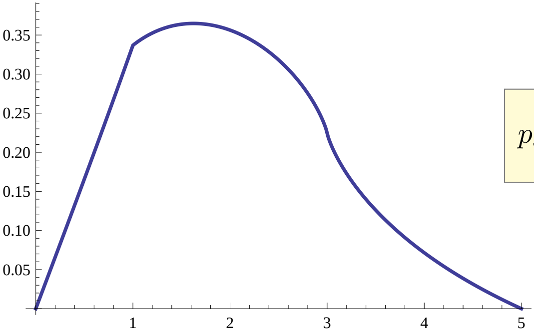
Direct combinatorial proof by Zagier.

p_5 — starting startlingly straight



$$p_5(x) = \int_0^\infty xtJ_0(xt)J_0^5(t)\,\mathrm{d}t$$

> *... the graphical construction, however carefully reinvestigated, did not permit of our considering the curve to be anything but a* **straight** *line... Even if it is not absolutely true, it exemplifies the extraordinary power of such integrals of J products to give extremely close approximations to such simple forms as horizontal lines.*
>
> **Karl Pearson**, *1906*



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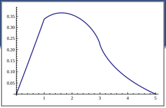
>
> **Karl Pearson**, *1906*

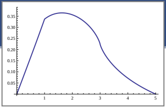
$$p_5(x) = 0.32993x + 0.0066167x^3 + 0.00026233x^5 + 0.000014119x^7 + O(x^9)$$

What we know about p_5



- $W_5(s)$ has simple poles at $-2k-2$ with residue $r_{5,k}$
- Hence: $p_5(x) = \sum_{k=0}^{\infty} r_{5,k}\, x^{2k+1}$

What we know about p_5



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- Hence: $p_5(x) = \sum_{k=0}^{\infty} r_{5,k}\, x^{2k+1}$

Surprising bonus of the modularity of p_4

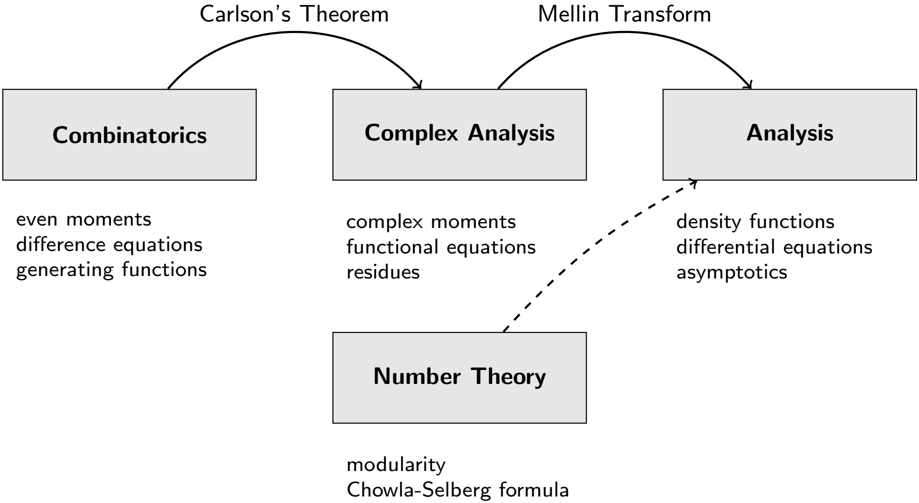
$$r_{5,0} = p_4(1) = \frac{\sqrt{5}}{40} \frac{\Gamma(\frac{1}{15})\Gamma(\frac{2}{15})\Gamma(\frac{4}{15})\Gamma(\frac{8}{15})}{\pi^4}$$

$$r_{5,1} \overset{?}{=} \frac{13}{225} r_{5,0} - \frac{2}{5\pi^4} \frac{1}{r_{5,0}}$$

- Other residues given recursively
- p_5 solves the DE

$$\Big[x^6(\theta+1)^4 - x^4(35\theta^4 + 42\theta^2 + 3) + x^2(259(\theta-1)^4 + 104(\theta-1)^2) - (15(\theta-3)(\theta-1))^2\Big] \cdot p_5(x) = 0$$

Summary of the ingredients



THANK YOU!

- Slides for this talk will be available from my website:
  `http://arminstraub.com/talks`

J. Borwein, A. Straub, J. Wan, W. Zudilin
Densities of short uniform random walks
Canadian Journal of Mathematics — to appear