

SHORT WALK ADVENTURES

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To the memory of Jon Borwein, who convinced us that a short walk can be adventurous

ABSTRACT. We review recent development of short uniform random walks, with a focus on its connection to (zeta) Mahler measures and modular parametrisation of the density functions. Furthermore, we extend available “probabilistic” techniques to cover a variation of random walks and reduce some three-variable Mahler measures, which are conjectured to evaluate in terms of L -values of modular forms, to hypergeometric form.

0. INTRODUCTION

At some stages of our careers we were approached by Jon Borwein to collaborate on a theme that sounded rather off topic to us, who had interests in number theory, combinatorics and related special functions. Somewhat unexpectedly, the theme has become a remarkable research project with several outcomes (including [8, 9, 10], to list a few), a project which we continue to enjoy after the sudden loss of Jon. . . This note serves as a summary to our recent discoveries that certain “probabilistic” techniques apply usefully to tackling difficult problems on the border of analysis, number theory and differential equations; in particular, in evaluating multi-variable Mahler measures. Our principal novelties are given in Theorems 1–3; these include hypergeometric reduction of the Mahler measures of the three-variable polynomials

$$1 + x_1 + x_2 + x_3 + x_2x_3 \quad \text{and} \quad (1 + x_1)^2 + x_2 + x_3,$$

as well as the (hypergeometric) factorisation of a related differential operator for the Apéry-like sequence

$$\sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k}, \quad \text{where } n = 0, 1, 2, \dots .$$

Echoing Jon’s “a short walk can be beautiful” [7], we add that “a short walk can be adventurous.”

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1. UNIFORM RANDOM WALKS

An N -step uniform random walk is a planar walk that starts at the origin and consists of N steps of length 1 each taken into a uniformly random direction. Let X_N be the distance to the origin after these N steps. The s -th moments $W_N(s)$ of X_N can be computed [10] via the formula

$$\begin{aligned} W_N(s) &= \int \cdots \int_{[0,1]^N} |e^{2\pi i\theta_1} + \cdots + e^{2\pi i\theta_N}|^s d\theta_1 \cdots d\theta_N \\ &= \int \cdots \int_{[0,1]^{N-1}} |1 + e^{2\pi i\theta_1} + \cdots + e^{2\pi i\theta_{N-1}}|^s d\theta_1 \cdots d\theta_{N-1}, \end{aligned}$$

and are related to the (probability) density function $p_N(x)$ of X_N via

$$W_N(s) = \int_0^\infty x^s p_N(x) dx = \int_0^N x^s p_N(x) dx.$$

That is, $p_N(x)$ can then be obtained as the inverse Mellin transform of $W_N(s-1)$. Finally, note that the even moments $W_3(2n)$ and $W_4(2n)$ (which are, clearly, positive integers) can be identified with the odd moments of $I_0(t)K_0(t)^2$ and $I_0(t)K_0(t)^3$, respectively, where $I_0(t)$ and $K_0(t)$ denote the modified Bessel functions of the first and second kind. Namely, for $n = 1, 2, \dots$ we have [5]

$$W_3(2n) = \frac{3^{2n+3/2}}{\pi 2^{2n} n!^2} \int_0^\infty t^{2n+1} I_0(t) K_0(t)^2 dt$$

and

$$W_4(2n) = \frac{4^{2n+2}}{\pi^2 n!^2} \int_0^\infty t^{2n+1} I_0(t) K_0(t)^3 dt.$$

2. ZETA MAHLER MEASURES

For a non-zero Laurent polynomial $P(x_1, \dots, x_N) \in \mathbb{C}[x_1^{\pm 1}, \dots, x_N^{\pm 1}]$, its zeta Mahler measure [3] is defined by

$$Z(P; s) = \int \cdots \int_{[0,1]^N} |P(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_N})|^s d\theta_1 \cdots d\theta_N,$$

and its logarithmic Mahler measure is

$$m(P) = \left. \frac{dZ(P; s)}{ds} \right|_{s=0} = \int \cdots \int_{[0,1]^N} \log |P(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_N})| d\theta_1 \cdots d\theta_N.$$

A straightforward comparison of the two definitions reveals that

$$W_N(s) = Z(x_1 + \cdots + x_N; s) = Z(1 + x_1 + \cdots + x_{N-1}; s)$$

and

$$W'_N(0) = m(x_1 + \cdots + x_N) = m(1 + x_1 + \cdots + x_{N-1}) = \int_0^N p_N(x) \log x dx, \quad (1)$$

where the derivative is with respect to s . The latter Mahler measures are known as linear Mahler measures. The evaluations

$$W_2'(0) = 0, \quad W_3'(0) = L'(\chi_{-3}; -1) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}; 2), \quad W_4'(0) = -14\zeta'(-2) = \frac{7\zeta(3)}{2\pi^2},$$

are known [22], while the following conjectural evaluations, due to Rodriguez-Villegas [12] and verified to several hundred digits [4], remain open:

$$W_5'(0) \stackrel{?}{=} -L'(f_3; -1) = 6 \left(\frac{\sqrt{15}}{2\pi} \right)^5 L(f_3; 4),$$

$$W_6'(0) \stackrel{?}{=} -8L'(f_4; -1) = 3 \left(\frac{\sqrt{6}}{\pi} \right)^6 L(f_4; 5),$$

where

$$f_3(\tau) = \eta(\tau)^3 \eta(15\tau)^3 + \eta(3\tau)^3 \eta(5\tau)^3 \quad \text{and} \quad f_4(\tau) = \eta(\tau)^2 \eta(2\tau)^2 \eta(3\tau)^2 \eta(6\tau)^2$$

are cusp eigenforms of weight 3 and 4, respectively. Here and in what follows, Dedekind's eta function

$$\eta(\tau) = q^{1/24} \prod_{m=1}^{\infty} (1 - q^m) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(6n+1)^2/24}, \quad \text{where } q = e^{2\pi i\tau},$$

serves as a principal constructor of modular forms and functions. No similar formulae are known for $W'_N(0)$ when $N \geq 7$, though the story continues at a different level—see [13, 28, 29] for details.

3. GENERIC TWO-STEP RANDOM WALKS

Let X_1 and X_2 be two (sufficiently nice, independent) random variables on $[0, \infty)$ with probability density $p_1(x)$ and $p_2(x)$, respectively, and let θ_1 and θ_2 be uniformly distributed on $[0, 1]$. Then $X = e^{2\pi i\theta_1} X_1 + e^{2\pi i\theta_2} X_2$ describes a two-step random walk in the plane with a first step of length X_1 and a second step of length X_2 . As in [9, eq. (3-3)], an application of the cosine rule shows that the s -th moment of $|X|$ is

$$W(s) = \mathbf{E}(|X|^s) = \int_0^\infty \int_0^\infty g_s(x, y) p_1(x) p_2(y) dx dy,$$

where

$$g_s(x, y) = \frac{1}{\pi} \int_0^\pi (x^2 + y^2 + 2xy \cos \theta)^{s/2} d\theta.$$

Observe that

$$\left. \frac{dg_s(x, y)}{ds} \right|_{s=0} = \frac{1}{\pi} \int_0^\pi \log \sqrt{x^2 + y^2 + 2xy \cos \theta} d\theta = \max\{\log |x|, \log |y|\},$$

so that, in particular,

Lemma 1. *We have*

$$W'(0) = \mathbf{E}(\log |X|) = \int_0^\infty \int_0^\infty p_1(x) p_2(y) \max\{\log x, \log y\} dy dx.$$

Alternative equivalent expressions, that will be useful in what follows, include

$$\begin{aligned} \mathbb{E}(\log |X|) &= \int_0^\infty \int_0^x p_1(x)p_2(y) \log x \, dy \, dx + \int_0^\infty \int_x^\infty p_1(x)p_2(y) \log y \, dy \, dx \\ &= \mathbb{E}(\log X_1) + \int_0^\infty \int_x^\infty p_1(x)p_2(y)(\log y - \log x) \, dy \, dx \\ &= \mathbb{E}(\log X_2) + \int_0^\infty \int_0^x p_1(x)p_2(y)(\log x - \log y) \, dy \, dx. \end{aligned} \quad (2)$$

4. LINEAR MAHLER MEASURES

Let N, M be integers such that $N > M > 0$. By decomposing an N -step random walk into two walks with $N - M$ and M steps, and applying Lemma 1 in the form (2), we find that

$$W'_N(0) = W'_M(0) + \int_0^{N-M} p_{N-M}(x) \left(\int_0^x p_M(y)(\log x - \log y) \, dy \right) dx.$$

This formula, together with known formulae for the densities [10], like $p_1(x) = \delta(x - 1)$ (the Dirac delta function) and $p_2(x) = 2/(\pi\sqrt{4 - x^2})$ for $0 < x < 2$, allows one to produce new expressions for linear Mahler measures. Indeed, taking $M = 1$ we get

$$W'_N(0) = \int_1^{N-1} p_{N-1}(x) \log x \, dx \quad (3)$$

(which can be also derived using Jensen's formula), while $M = 2$ results in

$$W'_N(0) = \int_2^{N-2} p_{N-2}(x) \log x \, dx + \frac{1}{\pi} \int_0^2 p_{N-2}(x) x \cdot {}_3F_2 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| \frac{x^2}{4} \right) dx \quad (4)$$

(see also [18, eq. (2.1)]). Here, and in what follows, the hypergeometric notation

$${}_mF_{m-1} \left(\begin{matrix} a_1, a_2, \dots, a_m \\ b_2, \dots, b_m \end{matrix} \middle| z \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_m)_n}{(b_2)_n \cdots (b_m)_n} \frac{z^n}{n!}$$

is used, where

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} a(a+1) \cdots (a+n-1), & \text{for } n \geq 1, \\ 1, & \text{for } n = 0, \end{cases}$$

denotes the Pochhammer symbol (the rising factorial). Note that we deduce (4) from

$$\int_0^x p_2(y)(\log x - \log y) \, dy = \frac{x}{\pi} \cdot {}_3F_2 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| \frac{x^2}{4} \right),$$

which is valid if $0 \leq x \leq 2$.

Equations (3) and (4) and the formula

$$p_4(x) = \frac{2\sqrt{16-x^2}}{\pi^2 x} \operatorname{Re} {}_3F_2 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{5}{6}, \frac{7}{6} \end{matrix} \middle| \frac{(16-x^2)^3}{108x^4} \right)$$

obtained in [10, Theorem 4.9], provide the formulae

$$W'_5(0) = \frac{7\zeta(3)}{2\pi^2} - \frac{1}{\pi^2} \int_0^1 \sqrt{16-x^2} \operatorname{Re} {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \mid \frac{(16-x^2)^3}{108x^4}\right) d(\log^2 x)$$

and

$$\begin{aligned} W'_6(0) &= \frac{7\zeta(3)}{2\pi^2} - \frac{1}{\pi^2} \int_0^2 \sqrt{16-x^2} \operatorname{Re} {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \mid \frac{(16-x^2)^3}{108x^4}\right) d(\log^2 x) \\ &\quad + \frac{2}{\pi^3} \int_0^2 \sqrt{16-x^2} \operatorname{Re} {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \mid \frac{(16-x^2)^3}{108x^4}\right) \cdot {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \mid \frac{x^2}{4}\right) dx. \end{aligned}$$

These *single* integrals can be used to numerically confirm the conjectural evaluations of $W'_5(0)$ and $W'_6(0)$.

A similar application of Lemma 1, upon decomposing a 6-step walk into two walks with 3 steps, yields the alternative reduction

$$W'_6(0) = 2 \int_0^3 p_3(x) \log x \left(\int_0^x p_3(y) dy \right) dx, \quad (5)$$

where [10]

$$p_3(x) = \frac{2\sqrt{3}x}{\pi(3+x^2)} \cdot {}_2F_1\left(\frac{1}{3}, \frac{2}{3} \mid \frac{x^2(9-x^2)^2}{(3+x^2)^3}\right).$$

We discuss this formula further in Section 5.

Finally, we mention that equation (3) and a modular parametrisation of $p_4(x)$ (which we indicate in Section 6) were independently cast in [21] to produce a double L -value expression for $W'_5(0)$.

5. MODULAR PARAMETRISATION OF $p_3(x)$ AND RELATED FORMULAE

Note that formula (5) can be written as

$$W'_6(0) = \int_0^3 \log x d(P_3(x)^2) = \log 3 - \int_0^3 P_3(x)^2 \frac{dx}{x},$$

featuring the cumulative density function

$$P_3(x) = \int_0^x p_3(y) dy.$$

The related modular parametrisation of $p_3(x)$ is given by

$$x = x(\tau) = 3 \frac{\eta(\tau)^2 \eta(6\tau)^4}{\eta(2\tau)^4 \eta(3\tau)^2} : (i\infty, 0) \rightarrow (0, 3),$$

so that

$$p_3(x) = \frac{2\sqrt{3}}{\pi} \frac{\eta(2\tau)^2 \eta(6\tau)^2}{\eta(\tau) \eta(3\tau)}, \quad dx = 3\pi i \frac{\eta(\tau)^6 \eta(3\tau)^2 \eta(6\tau)^2}{\eta(2\tau)^6} d\tau$$

and

$$P_3(x) = 6i\sqrt{3} \int_{i\infty}^\tau \frac{\eta(\tau)^5 \eta(3\tau) \eta(6\tau)^4}{\eta(2\tau)^4} d\tau$$

is the anti-derivative of a weight 3 holomorphic Eisenstein series

$$\frac{\eta(\tau)^5\eta(3\tau)\eta(6\tau)^4}{\eta(2\tau)^4} = E_{3,\chi_{-3}}(\tau) - 8E_{3,\chi_{-3}}(2\tau),$$

where

$$E_{3,\chi_{-3}}(\tau) = \frac{\eta(3\tau)^9}{\eta(\tau)^3} = \sum_{m,n=1}^{\infty} \left(\frac{-3}{m}\right) n^2 q^{mn}, \quad \chi_{-3}(m) = \left(\frac{-3}{m}\right) = \frac{e^{2\pi i m/3} - e^{-2\pi i m/3}}{i\sqrt{3}}.$$

Though the anti-derivative $P_3(x)$,

$$\begin{aligned} P_3(x) &= \frac{3\sqrt{3}}{\pi} \left(\sum_{m,n=1}^{\infty} \left(\frac{-3}{m}\right) \frac{n}{m} q^{mn} - 4 \sum_{m,n=1}^{\infty} \left(\frac{-3}{m}\right) \frac{n}{m} q^{2mn} \right) \\ &= \frac{9i}{\pi} \log \prod_{n=1}^{\infty} \left(\frac{(1 - e^{2\pi i/3} q^{2n})^4 (1 - e^{-2\pi i/3} q^n)}{(1 - e^{-2\pi i/3} q^{2n})^4 (1 - e^{2\pi i/3} q^n)} \right)^n, \end{aligned}$$

is not considered to be sufficiently “natural”, it shows up as the elliptic dilogarithm thanks to Bloch’s formula; see [15, 17] for the details. Note that

$$E_{3,\chi_{-3}}\left(-\frac{1}{3\tau}\right) = \frac{i\tau^3}{3\sqrt{3}} \tilde{E}_{3,\chi_{-3}}(\tau), \quad \tilde{E}_{3,\chi_{-3}}(\tau) = \frac{\eta(\tau)^9}{\eta(3\tau)^3} = 1 - 9 \sum_{m,n=1}^{\infty} \left(\frac{-3}{n}\right) n^2 q^{mn};$$

and, in addition, we have

$$\begin{aligned} \frac{1}{2\pi i} \frac{dx/d\tau}{x} &= \frac{1}{2} \left(\frac{\eta(\tau)^2\eta(3\tau)^2}{\eta(2\tau)\eta(6\tau)} \right)^2 = \frac{1}{18} (E_{1,\chi_{-3}}(\tau) - 4E_{1,\chi_{-3}}(4\tau))^2 \\ &= \frac{1}{54\tau^2} \left(E_{1,\chi_{-3}}\left(-\frac{1}{12\tau}\right) - E_{1,\chi_{-3}}\left(-\frac{1}{3\tau}\right) \right)^2, \end{aligned}$$

where

$$E_{1,\chi_{-3}}(\tau) = 1 + 6 \sum_{m,n=1}^{\infty} \left(\frac{-3}{m}\right) q^{mn}.$$

6. MODULAR COMPUTATION FOR $W'_5(0)$ AND $W'_6(0)$

As (partly) shown in [10] the density $p_4(x)$ can be parameterised as follows (we make a shift of τ by half):

$$p_4(x(\tau)) = -\operatorname{Re} \left(\frac{2i(1 + 6\tau + 12\tau^2)}{\pi} p(\tau) \right),$$

where

$$p(\tau) = \frac{\eta(2\tau)^4\eta(6\tau)^4}{\eta(\tau)\eta(3\tau)\eta(4\tau)\eta(12\tau)} \quad \text{and} \quad x(\tau) = \left(\frac{2\eta(\tau)\eta(3\tau)\eta(4\tau)\eta(12\tau)}{\eta(2\tau)^2\eta(6\tau)^2} \right)^3.$$

The path for τ along the imaginary axis from 0 to $i/(2\sqrt{3})$ (or from $i\infty$ to $i/(2\sqrt{3})$) corresponds to x ranging from 0 to 2, while the path from $i/(2\sqrt{3})$ to $-1/4 + i/(4\sqrt{3})$ along the arc centred at 0 corresponds to the real range (2, 4) for x . (The arc admits

the parametrisation $\tau = e^{\pi i \theta} / (2\sqrt{3})$, $1/2 < \theta < 5/6$.) Note that $x(i/(2\sqrt{15})) = 1$ and

$$p_4(x(\tau)) = \begin{cases} -\frac{2i \cdot 6\tau}{\pi} p(\tau), & \text{for } \tau \text{ on the imaginary axis,} \\ -\frac{2i(1 + 6\tau + 12\tau^2)}{\pi} p(\tau), & \text{for } \tau \text{ on the arc,} \end{cases}$$

and

$$-\frac{2i(1 + 6\tau + 12\tau^2)}{\pi} p(\tau) = \frac{2\sqrt{16 - x^2}}{\pi^2 x} \cdot {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \mid \frac{(16 - x^2)^3}{108x^4}\right)$$

(this is a general form of [10, Theorem 4.9]). Formulas (1), (3) and (4) reduce the conjectural evaluations of $W'_5(0)$ and $W'_6(0)$ to the following ones:

$$\frac{7\zeta(3)}{2\pi^2} + L'(f_3; -1) \stackrel{?}{=} \frac{12}{\pi} \int_0^{1/(2\sqrt{15})} yp(iy) \log x(iy) dx(iy)$$

and

$$\begin{aligned} \frac{7\zeta(3)}{2\pi^2} + 8L'(f_4; -1) &\stackrel{?}{=} \frac{12}{\pi} \int_0^{1/(2\sqrt{3})} yp(iy) \log x(iy) dx(iy) \\ &- \frac{12}{\pi^2} \int_0^{1/(2\sqrt{3})} yp(iy)x(iy) \cdot {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \mid \frac{x(iy)^2}{4}\right) dx(iy). \end{aligned}$$

Furthermore, note that the Atkin–Lehner involutions $w_{12}: \tau \mapsto -1/(12\tau)$ and $w_6: \tau \mapsto (6\tau - 5)/(12\tau - 6)$ act on the modular function $x(\tau)$ as follows: $x(w_{12}\tau) = x(\tau)$ and $x(w_6\tau) = -8/x(\tau)$, and we also have $p(w_{12}\tau) = -\tau^2 p(\tau)$. Thus, the change of variable $y \mapsto 1/(12y)$ leads to

$$\int_0^{1/(2\sqrt{3})} yp(iy) \log x(iy) dx(iy) = - \int_{1/(2\sqrt{3})}^{\infty} yp(iy) \log x(iy) dx(iy).$$

7. MAHLER MEASURES RELATED TO A VARIATION OF RANDOM WALK

In [21] the Mahler measures $m(1 + x_1 + x_2)$ and $m(1 + x_1 + x_2 + x_3)$ are computed using the modular parametrisations of

$$\sum_{n=0}^{\infty} W_3(2n)z^n = \sum_{n=0}^{\infty} \text{CT}((1 + x_1 + x_2)(1 + x_1^{-1} + x_2^{-1}))^n z^n$$

and

$$\sum_{n=0}^{\infty} W_4(2n)z^n = \sum_{n=0}^{\infty} \text{CT}((1 + x_1 + x_2 + x_3)(1 + x_1^{-1} + x_2^{-1} + x_3^{-1}))^n z^n,$$

where $\text{CT}(L)$ denotes the constant term of a Laurent polynomial $L \in \mathbb{Z}[x_1^{\pm}, x_2^{\pm}, \dots]$. Note that the Picard–Fuchs linear differential equations for the two generating functions give rise to the ones for the densities $p_3(x)$ and $p_4(x)$ together with their explicit hypergeometric and modular expressions (see [10, eq. (3.2) and Remark 4.10]), though it remains unclear whether the latter information can be used to compute $W'_N(0)$ in (1) for $N = 3, 4$. This is itself an interesting question to not only assist in

computing of $W'_N(0)$ for $N > 4$ but also in relation with another famous conjecture of Boyd:

$$m(1 + x_1 + x_2 + x_3 + x_2x_3) \stackrel{?}{=} 2L'(f_2; -1) = \frac{15^2}{4\pi^4} L(f_2; 3) = 0.4839979734 \dots, \quad (6)$$

where $f_2(\tau) = \eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau)$.

In analogy with the case of linear Mahler measures, we define

$$\begin{aligned} \widetilde{W}(s) &= \iiint_{[0,1]^3} |1 + e^{2\pi i\theta_1} + e^{2\pi i\theta_2} + e^{2\pi i\theta_3} + e^{2\pi i(\theta_2+\theta_3)}|^s d\theta_1 d\theta_2 d\theta_3 \\ &= Z(1 + x_1 + x_2 + x_3 + x_2x_3; s) \end{aligned}$$

as the s -th moment of a random 5-step walk for which the direction of the final step is completely determined by the two previous steps. Then the even moments

$$\begin{aligned} \widetilde{W}(2n) &= \text{CT}((1 + x_1 + x_2 + x_3 + x_2x_3)(1 + x_1^{-1} + x_2^{-1} + x_3^{-1} + (x_2x_3)^{-1}))^n \\ &= \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k} \end{aligned}$$

satisfy a complicated recurrence equation which is equivalent to a Picard–Fuchs differential equation of order 4. The latter splits into the tensor product of two differential equations of order 2 and, with some effort, we obtain the following result.

Theorem 1. *We have*

$$\begin{aligned} \sum_{n=0}^{\infty} \widetilde{W}(2n) \left(\frac{t}{(4+t)(1+4t)} \right)^n \\ = \frac{(4+t)(1+4t)}{4(1+4t+t^2)} {}_2F_1\left(\frac{1}{2}, \frac{1}{2} \middle| 1 \middle| \frac{t(4+t)}{1+4t+t^2}\right) \cdot {}_2F_1\left(\frac{1}{2}, \frac{1}{2} \middle| 1 \middle| \frac{t^2}{1+4t+t^2}\right) \end{aligned}$$

and, more generally,

$$\begin{aligned} \frac{b}{(b+t)(1+bt)} \sum_{n=0}^{\infty} \left(\frac{t}{(b+t)(1+bt)} \right)^n \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k} \left(\frac{b}{4}\right)^{2k} \\ = {}_2F_1\left(\frac{1}{2}, \frac{1}{2} \middle| 1 \middle| -t(b+t)\right) \cdot \frac{1}{(1+bt)^{1/2}} {}_2F_1\left(\frac{1}{2}, \frac{1}{2} \middle| 1 \middle| -\frac{t^2}{1+bt}\right) \\ = \frac{1}{1+bt+t^2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2} \middle| 1 \middle| \frac{t(b+t)}{1+bt+t^2}\right) \cdot {}_2F_1\left(\frac{1}{2}, \frac{1}{2} \middle| 1 \middle| \frac{t^2}{1+bt+t^2}\right). \end{aligned}$$

Proof. Once a factorisation of this type is written down, it is a computational routine to prove it. In other words, a principal issue is discovering such a formula rather than proving it. Our original discovery of Theorem 1 involved a lot of experimental mathematics; however, we later realised that it is deducible from known formulae

as follows:

$$\begin{aligned}
\sum_{n=0}^{\infty} z^n \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k}^2 x^k &= \sum_{k=0}^{\infty} \binom{2k}{k}^2 x^k \sum_{m=0}^{\infty} \binom{k+m}{k}^2 z^{k+m} \\
&= \sum_{k=0}^{\infty} \binom{2k}{k}^2 (xz)^k {}_2F_1 \left(\begin{matrix} k+1, k+1 \\ 1 \end{matrix} \middle| z \right) \\
&= \sum_{k=0}^{\infty} \binom{2k}{k}^2 \frac{(xz)^k}{(1-z)^{k+1}} {}_2F_1 \left(\begin{matrix} -k, k+1 \\ 1 \end{matrix} \middle| -\frac{z}{1-z} \right) \\
&= \frac{1}{1-z} \sum_{k=0}^{\infty} \binom{2k}{k}^2 \left(\frac{xz}{1-z} \right)^k \cdot P_k \left(\frac{1+z}{1-z} \right),
\end{aligned}$$

where P_k denotes the k -th Legendre polynomial, and the latter generating function is a particular instance of the Bailey–Brafman formula [14, 32]. \square

We remark that, using the general Bailey–Brafman formula and its generalisation from [27], the proof above extends to the factorisation of the two-variable generating functions

$$\sum_{n=0}^{\infty} z^n \sum_{k=0}^n \binom{n}{k}^2 \frac{(s)_k (1-s)_k}{k!^2} x^k$$

as well as of

$$\sum_{n=0}^{\infty} z^n \sum_k \binom{n}{2k}^2 \binom{2k}{k}^2 x^k \quad \text{and} \quad \sum_{n=0}^{\infty} z^n \sum_k \binom{n}{3k}^2 \frac{(3k)!}{k!^3} x^k,$$

and even of

$$\sum_{n=0}^{\infty} z^n \sum_{k=0}^n \binom{n}{k}^2 u_k x^k$$

for an Apéry-like sequence u_0, u_1, u_2, \dots .

Furthermore, we expect that Theorem 1 can lead to a hypergeometric expression for the density function $\tilde{p}(x)$ (piecewise analytic, with finite support on the interval $0 < x < 5$), which is the inverse Mellin transform of $\widetilde{W}(s-1)$, hence to the Mahler measure evaluation

$$\mathfrak{m}(1 + x_1 + x_2 + x_3 + x_2x_3) = \widetilde{W}'(0) = \int_0^{\infty} \tilde{p}(x) \log x \, dx = \int_0^5 \tilde{p}(x) \log x \, dx.$$

On the other hand, the reduction technique of Sections 3 and 4 suggests a different approach to computing $\widetilde{W}'(0)$, resulting in the following hypergeometric evaluation of the Mahler measure.

Theorem 2. *We have*

$$\mathfrak{m}(1 + x_1 + x_2 + x_3 + x_2x_3) = -\frac{1}{2\pi} \int_0^1 {}_2F_1 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \middle| 1 - \frac{x^2}{16} \right) \log x \, dx.$$

Proof. Define a related density $\widehat{p}(x)$ by

$$\begin{aligned} \int_0^4 x^s \widehat{p}(x) dx &= \widehat{W}(s) = \iint_{[0,1]^2} |1 + e^{2\pi i\theta_2} + e^{2\pi i\theta_3} + e^{2\pi i(\theta_2+\theta_3)}|^s d\theta_2 d\theta_3 \\ &= W_2(s)^2 = \frac{\Gamma(1+s)^2}{\Gamma(1+s/2)^4}. \end{aligned}$$

By an application of the Mellin transform calculus, we find that, for $0 < x < 4$,

$$\widehat{p}(x) = \frac{1}{2\pi} \cdot {}_2F_1\left(\frac{1}{2}, \frac{1}{2} \middle| 1 - \frac{x^2}{16}\right).$$

Then it follows from Lemma 1 that

$$\widetilde{W}'(0) = \int_1^4 \widehat{p}(x) \log x dx = - \int_0^1 \widehat{p}(x) \log x dx,$$

where we use the evaluation

$$\int_0^4 \widehat{p}(x) \log x dx = m(1 + x_2 + x_3 + x_2x_3) = m(1 + x_2) + m(1 + x_3) = 0. \quad \square$$

A related computation

$$m(1 + bx_1 + x_2 + x_3 + x_2x_3) = \log b + \frac{8}{\pi^2} \int_b^4 \frac{\arccos(b/x) \log(x/(2\sqrt{b}))}{\sqrt{16 - x^2}} dx$$

valid for $0 < b \leq 4$ was given by J. Wan [25]; he also pointed out that $m(1 + bx_1 + x_2 + x_3 + x_2x_3) = \log b$ for $b > 4$ follows from Jensen's formula.

The left-hand side of another Mahler measure conjecture [12]

$$m((1 + x_1)^2 + x_2 + x_3) \stackrel{?}{=} 2L'(\tilde{f}_2; -1) = \frac{144}{\pi^4} L(\tilde{f}_2; 3) = 1.4051310125\dots,$$

where $\tilde{f}_2(\tau) = \eta(2\tau)\eta(4\tau)\eta(6\tau)\eta(12\tau)$ is a cusp form of level 24, can be treated by a similar reduction, using that the densities for $(1 + x_1)^2$ and $x_2 + x_3$ are $p_2(t^{1/2})/(2t^{1/2})$ on $[0, 4]$ and $p_2(t)$ on $[0, 2]$, respectively. The final result is the elegant formula

$$m((1 + x_1)^2 + x_2 + x_3) = \frac{4G}{\pi} + \frac{4}{\pi^2} \int_0^1 \arcsin(1 - x) \arcsin x \frac{dx}{x}, \quad (7)$$

where G is Catalan's constant, and, with some further work, we can express the right-hand side hypergeometrically.

Theorem 3. *We have*

$$\begin{aligned} m((1 + x_1)^2 + x_2 + x_3) &= \frac{16\Gamma(\frac{3}{4})^2}{\pi^{5/2}} {}_5F_4\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4} \middle| \frac{1}{4}\right) \\ &\quad + \frac{\Gamma(\frac{1}{4})^2}{27\pi^{5/2}} {}_5F_4\left(\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{5}{4}, \frac{5}{4} \middle| \frac{1}{4}\right). \end{aligned}$$

Proof. Notice that, for $0 < x < 1$,

$$\arcsin(1-x) = \frac{\pi}{2} - \arccos(1-x) = \frac{\pi}{2} - \sqrt{2x} {}_2F_1\left(\frac{1}{2}, \frac{1}{2} \middle| \frac{x}{2}\right),$$

and that, for $n > -1/2$,

$$\int_0^1 x^{n-1/2} \arcsin x \, dx = \frac{\sqrt{\pi}}{2n+1} \left(\sqrt{\pi} - \frac{\Gamma(\frac{n}{2} + \frac{3}{4})}{\Gamma(\frac{n}{2} + \frac{5}{4})} \right).$$

Therefore,

$$\begin{aligned} \int_0^1 \arcsin(1-x) \arcsin x \frac{dx}{x} &= \frac{\pi}{2} \int_0^1 \arcsin x \frac{dx}{x} - \pi\sqrt{2} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{n! (\frac{3}{2})_n (2n+1)} \frac{1}{2^n} \\ &\quad + \sqrt{2\pi} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2 \Gamma(\frac{n}{2} + \frac{3}{4})}{n! (\frac{3}{2})_n (2n+1) \Gamma(\frac{n}{2} + \frac{5}{4})} \frac{1}{2^n}. \end{aligned}$$

From this and (7) we deduce

$$\begin{aligned} m((1+x_1)^2 + x_2 + x_3) &= \frac{4G}{\pi} + \log 2 - \frac{4\sqrt{2}}{\pi} {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \middle| \frac{1}{2}\right) \\ &\quad + \frac{16\sqrt{2}\Gamma(\frac{3}{4})}{\pi^{3/2}\Gamma(\frac{1}{4})} {}_5F_4\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4} \middle| \frac{1}{4}\right) + \frac{\sqrt{2}\Gamma(\frac{1}{4})}{27\pi^{3/2}\Gamma(\frac{3}{4})} {}_5F_4\left(\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{5}{4}, \frac{5}{4} \middle| \frac{1}{4}\right). \end{aligned}$$

It remains to use

$$G + \frac{1}{4}\pi \log 2 = \sqrt{2} {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \middle| \frac{1}{2}\right)$$

(see [1, Entry 30]) and $\Gamma(\frac{1}{4})\Gamma(\frac{3}{4}) = \pi\sqrt{2}$. □

8. CONCLUSION

A goal of this final section is to highlight relevance for and links with other research and open problems.

The (hypergeometric) factorisation in Theorem 1 and similar results outlined after its proof are part of a general phenomenon of arithmetic differential equations of order 4. These are the first instances “beyond modularity” in the sense that arithmetic differential equations of order 2 and 3 are always supplied by modular parametrisation. In order 4, we have to distinguish two particular novel situations (though our knowledge about either is imperfect and incomplete): (the Zariski closure of) the monodromy group is the orthogonal group $O_4 \simeq O_{2,2}$ of dimension 6 or the symplectic group Sp_4 of dimension 10. The example given in Theorem 1 corresponds to the first (orthogonal) situation: on the level of Lie groups, $O_{2,2}$ can be realised as the tensor product of two copies of SL_2 (or GL_2). There is a limited amount of further examples of this type [19, 27, 31] though we expect that all underlying Picard–Fuchs differential equations with such monodromy can be respresented as tensor products of two arithmetic differential equations of order 2. There is a natural hypergeometric production of such orthogonal cases using Orr-type formulae (see [16, 26]) but there are plenty of other cases coming from classical work of W.N. Bailey and its

recent generalisations [27, 32]. Many such cases, mostly forecast by Sun [23], are still awaiting their explicit factorisation.

More in the direction of three-variable Mahler measure, the conjectural evaluation in (6) and Theorem 3 brings us to the expectation

$$-\frac{1}{2\pi} \int_0^1 {}_2F_1\left(\frac{1}{2}, \frac{1}{2} \middle| 1 - \frac{x^2}{16}\right) \log x \, dx \stackrel{?}{=} 2L'(f_2; -1).$$

This one highly resembles the evaluation

$$\frac{1}{2} \int_0^1 {}_2F_1\left(\frac{1}{2}, \frac{1}{2} \middle| \frac{x^2}{16}\right) \, dx = \frac{1}{2} \cdot {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \middle| \frac{1}{16}\right) = 2L'(f_2; 0)$$

established in [20]. The related modular parametrisation

$$x = x(\tau) = 16 \left(\frac{\eta(\tau)\eta(4\tau)^2}{\eta(2\tau)^3} \right)^4$$

corresponds to

$$1 - \frac{x^2}{16} = \left(\frac{\eta(\tau)^2\eta(4\tau)}{\eta(2\tau)^3} \right)^8,$$

$$F\left(\frac{x^2}{16}\right) = \frac{\eta(2\tau)^{10}}{\eta(\tau)^4\eta(4\tau)^4} \quad \text{and} \quad F\left(1 - \frac{x^2}{16}\right) = -2i\tau F\left(\frac{x^2}{16}\right),$$

where F denotes the corresponding ${}_2F_1$ hypergeometric series. Note that x ranges from 0 to 4 when τ runs from $i\infty$ to 0 along the imaginary axis; however, the point $\tau_0 = i0.8774376613482\dots$, at which $x(\tau_0) = 1$, is not a quadratic irrationality.

The expression in Theorem 3 is somewhat different from the one in Theorem 2, and resembles the hypergeometric evaluation of the L -value

$$L'(\hat{f}_2; -1) = \frac{128}{\pi^4} L(\hat{f}_2; 3)$$

$$= \frac{\Gamma(\frac{1}{4})^2}{6\sqrt{2}\pi^{5/2}} {}_4F_3\left(1, 1, 1, \frac{1}{2} \middle| 1\right) + \frac{4\Gamma(\frac{3}{4})^2}{\sqrt{2}\pi^{5/2}} {}_4F_3\left(1, 1, 1, \frac{1}{2} \middle| 1\right)$$

$$+ \frac{\Gamma(\frac{1}{4})^2}{2\sqrt{2}\pi^{5/2}} {}_4F_3\left(1, 1, 1, \frac{1}{2} \middle| 1\right),$$

where $\hat{f}_2(\tau) = \eta(4\tau)^2\eta(8\tau)^2$ is a cusp form of level 32, obtained in [30, Theorem 3].

Finally, we remark that the integral

$$W'_3(0) = \int_0^3 \log x \, dP_3(x) = \log 3 - \int_0^3 P_3(x) \frac{dx}{x}$$

in the notation of Section 5, with $P_3(x)$ related to eta quotients, is visually linked to the following result in [6] (also discussed in greater generality in [2, 24])

$$\int_0^1 \frac{1}{9} \left(1 - \frac{\eta(\tau)^9}{\eta(3\tau)^3}\right) \frac{dq}{q} = \lim_{q \rightarrow 1^-} \sum_{m,n=1}^{\infty} \binom{-3}{n} \frac{n}{m} q^{mn} = L'(\chi_{-3}; -1).$$

However, apart from the fact that the two quantities coincide we could not find a direct link between the two integrals.

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