

# Interpolated sequences and critical $L$ -values of modular forms

Robert Osburn <sup>\*</sup>      Armin Straub <sup>†</sup>

June 13, 2018

## Abstract

Recently, Zagier expressed an interpolated version of the Apéry numbers for  $\zeta(3)$  in terms of a critical  $L$ -value of a modular form of weight 4. We extend this evaluation in two directions. We first prove that interpolations of Zagier's six sporadic sequences are essentially critical  $L$ -values of modular forms of weight 3. We then establish an infinite family of evaluations between interpolations of leading coefficients of Brown's cellular integrals and critical  $L$ -values of modular forms of odd weight.

## 1 Introduction

For  $x \in \mathbb{C}$ , consider the absolutely convergent series

$$A(x) = \sum_{k=0}^{\infty} \binom{x}{k}^2 \binom{x+k}{k}^2. \quad (1)$$

If  $x = n \in \mathbb{Z}_{\geq 0}$ , this series terminates at  $k = n$  and agrees with the well-known Apéry numbers  $A(n)$  for  $\zeta(3)$  [5], [29]. Let

$$f(\tau) = \sum_{n \geq 1} a_n q^n \in S_k(\Gamma_1(N)), \quad q = e^{2\pi i \tau},$$

be a cusp form of weight  $k$  and level  $N$ , and

$$L(f, s) := \sum_{n \geq 1} \frac{a_n}{n^s}$$

be the  $L$ -function for  $f$  defined for  $\operatorname{Re} s$  large. Following Deligne, we say that  $L(f, j)$  is a *critical  $L$ -value* if  $j \in \{1, 2, \dots, k-1\}$ . For a beautiful exposition concerning the importance of these numbers, see [22]. Zagier [39, (44)]

---

<sup>\*</sup>School of Mathematics and Statistics, University College Dublin, Belfield, Dublin 4, Ireland, *Email:* `robert.osburn@ucd.ie`

<sup>†</sup>Department of Mathematics and Statistics, University of South Alabama, 411 University Blvd N, Mobile, AL 36688, United States, *Email:* `straub@southalabama.edu`

recently showed that the interpolated Apéry numbers (1) are related to the critical  $L$ -value of a modular form of weight 4. Specifically, he proved the following intriguing identity:

$$A(-\frac{1}{2}) = \frac{16}{\pi^2} L(f, 2), \quad (2)$$

where

$$f(\tau) = \eta(2\tau)^4 \eta(4\tau)^4 = \sum_{n \geq 1} \alpha_n q^n \quad (3)$$

is the unique normalized Hecke eigenform in  $S_4(\Gamma_0(8))$  and  $\eta(\tau)$  is the Dedekind eta function. We note that, expressing the left-hand side as a hypergeometric series, the identity (2) was previously established by Rogers, Wan and Zucker [30]. The evaluation (2) can be seen as a continuous counterpart to the congruence

$$A(\frac{p-1}{2}) \equiv \alpha_p \pmod{p}, \quad (4)$$

which holds for primes  $p > 2$  and was established by Beukers [9], who further conjectured that the congruence (4) actually holds modulo  $p^2$ . This supercongruence was later proven by Ahlgren and Ono [2] using Gaussian hypergeometric series.

Zagier indicates that Golyshev predicted an evaluation of the form (2) based on motivic considerations and the connection of the Apéry numbers with the double covering of a related family of K3 surfaces. Here, we do not touch on these geometric considerations (see [39, Section 7] for further details), but only note that Golyshev's prediction further relies on the Tate conjecture, which remains open in the required generality. Identity (2), and similar ones to be explored in this paper, might therefore serve as evidence supporting the motivic philosophy and the Tate conjecture.

The goal of this paper is to extend Zagier's evaluation (2) in two directions. Firstly, in Section 2, we consider the six sporadic sequences that Zagier [38] obtained as integral solutions to Apéry-like second order recurrences. Based on numerical experiments, we observe that each of these sequences  $C_*(n)$  appears to satisfy congruences like (4) connecting them with the Fourier coefficients of a modular form  $f_*(\tau)$  of weight 3. For three of these sequences these congruences were shown by Stienstra and Beukers [35], while the other three congruences do not appear to have been recorded before. We prove two of these new cases, one using a general result of Verrill [36] and the other via  $p$ -adic analysis and comparison with another case. Our main objective is to show that in each case there is a version of Zagier's evaluation (2). For  $x \in \mathbb{C}$ , there is a natural interpolation  $C_*(x)$  of each sequence and the value  $C_*(-1/2)$  can in five of the six cases be expressed as  $\frac{\alpha}{\pi^2} L(f_*, 2)$  for  $\alpha \in \{6, 8, 12, 16\}$ . In the remaining case,  $C_*(x)$  has a pole at  $x = -1/2$ . Remarkably, the residue of that pole equals  $\frac{6}{\pi^2} L(f_*, 1)$ .

Secondly, Brown [13] recently introduced cellular integrals generalizing the linear forms used in Apéry's proof of the irrationality of  $\zeta(3)$  as well as many other constructions related to the irrationality of zeta values. These are linear

forms in multiple zeta values and their leading coefficients  $A_\sigma(n)$  are generalizations of the Apéry numbers. McCarthy and the authors [27] proved that, for a certain infinite family of these cellular integrals, the leading coefficients  $A_\sigma(n)$  satisfy congruences like (4) with Fourier coefficients of modular forms  $f_k(\tau)$  of odd weight  $k \geq 3$ . In Section 3, we review these facts and prove an analogue of Zagier’s evaluation (2) for all of these sequences. Finally, in Section 4, we conclude with several directions for future study.

## 2 Zagier’s sporadic sequences

### 2.1 The congruences and $L$ -value relations

In addition to  $A(n)$ , Apéry [5] introduced a second sequence which allowed him to reprove the irrationality of  $\zeta(2)$ . This sequence is the solution of the three-term recursion, for  $(a, b, c) = (11, 3, -1)$ ,

$$(n+1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2 u_{n-1}, \quad (5)$$

with initial conditions  $u_{-1} = 0$ ,  $u_0 = 1$ . Inspired by Beukers [10], Zagier [38] conducted a systematic search for parameters  $(a, b, c)$  which similarly result in integer solutions to the recurrence (5). After normalizing, and apart from degenerate cases, he discovered four hypergeometric, four Legendrian and six sporadic solutions. It remains an open question whether this list is complete. The six sporadic solutions are listed in Table 1. As in [38], we use the labels **A-F** and index the sequences accordingly.

For each of these sequences, a binomial sum representation is known. For instance, if  $(a, b, c) = (11, 3, -1)$ , then

$$C_D(n) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}. \quad (6)$$

Following Zagier’s approach for (1), we obtain an interpolation of a sporadic sequence by replacing the integer  $n$  in the binomial representation with a complex number  $x$  and extending the sum to all nonnegative integers  $k$ . Note that some care is needed for sequence **C** (see Example 2.7 in Section 2.4). The resulting interpolations are recorded in Table 1. We note that this construction depends on the binomial sum which is neither unique nor easily obtained from the recursion (5). The fact that we can relate the value of these interpolations at  $x = -1/2$  to critical  $L$ -values, as in Zagier’s evaluation (2), indicates that our choices are natural. We offer some more comments on these interpolations in Section 2.4.

Our first main result is the following.

**Theorem 2.1.** *Let  $C_*(n)$  be the sporadic sequence in Table 1 and  $f_*(\tau) = \sum_{n \geq 1} \gamma_{n,*} q^n$  be the weight 3, level  $N_*$  newform listed in Table 2 where the label  $*$  is **A**, **B**, **C**, **D** or **E**. Then, for all primes  $p > 2$ ,*

$$C_*\left(\frac{p-1}{2}\right) \equiv \gamma_{p,*} \pmod{p}. \quad (7)$$

$*$	$C_*(n)$	$C_*(x)$
$\mathbf{A}$	$\sum_{k=0}^n \binom{n}{k}^3$	$\sum_{k \geq 0} \binom{x}{k}^3$
$\mathbf{B}$	$\sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k 3^{n-3k} \binom{n}{3k} \frac{(3k)!}{k!^3}$	$\sum_{k \geq 0} (-1)^k 3^{x-3k} \binom{x}{3k} \frac{(3k)!}{k!^3}$
$\mathbf{C}$	$\sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k}$	$\operatorname{Re} {}_3F_2 \left[ \begin{matrix} -x, -x, 1/2 \\ 1, 1 \end{matrix} \middle  4 \right]$
$\mathbf{D}$	$\sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}$	$\sum_{k \geq 0} \binom{x}{k}^2 \binom{x+k}{k}$
$\mathbf{E}$	$\sum_{k=0}^n \binom{n}{k} \binom{2k}{k} \binom{2(n-k)}{n-k}$	$\sum_{k \geq 0} \binom{x}{k} \binom{2k}{k} \binom{2(x-k)}{x-k}$
$\mathbf{F}$	$\sum_{k=0}^n (-1)^k 8^{n-k} \binom{n}{k} C_{\mathbf{A}}(k)$	$\sum_{k \geq 0} (-1)^k 8^{x-k} \binom{x}{k} C_{\mathbf{A}}(k)$

Table 1: Zagier’s six sporadic sequences [38] and their interpolations

We note that the congruences (7) hold modulo  $p^2$  only for sequence  $\mathbf{D}$  [1]. Theorem 2.1 is known to be true for sequences  $\mathbf{C}$  and  $\mathbf{D}$  by work of Stienstra and Beukers [35], and we show in Section 2.2 that the congruences for sequence  $\mathbf{A}$  can be deduced from their work. The other three cases, including  $\mathbf{F}$ , appear not to have been considered before. We also show in Section 2.2 that a general result of Verrill [36] can be used to prove the modular congruences of Theorem 2.1 for sequences  $\mathbf{C}$  and  $\mathbf{E}$ . As she points out with sequence  $\mathbf{A}$ , the same approach does not apply in the other cases. Verrill indicates that the modular congruences for sequence  $\mathbf{A}$  can be explained by Atkin-Swinnerton-Dyer congruences [25, Section 5.8]. We expect that similar ideas can be applied to the case  $\mathbf{F}$ , for which we have numerically observed that Theorem 2.1 holds as well.

For our second main result, we have the following analogues of Zagier’s evaluation (2).

**Theorem 2.2.** *Let  $C_*(x)$  be the interpolated sequence in Table 1 and  $f_*(\tau)$  be the weight 3, level  $N_*$  newform listed in Table 2 where the label  $*$  is  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$ ,  $\mathbf{D}$  or  $\mathbf{F}$ . Then*

$$C_*(-\tfrac{1}{2}) = \frac{\alpha_*}{\pi^2} L(f_*, 2). \quad (8)$$

For sequence  $\mathbf{E}$ ,

$$\operatorname{res}_{x=-1/2} C_{\mathbf{E}}(x) = \frac{6}{\pi^2} L(f_{\mathbf{E}}, 1). \quad (9)$$

We prove Theorem 2.2 in Section 2.3. The proof for sequence  $\mathbf{F}$ , using a modular parametrization from [14], is due to Wadim Zudilin. We note from Table 2 that  $\alpha_*$  divides  $N_*$  in all cases except  $\mathbf{E}$ . It is natural to wonder if a uniform explanation can be given for this observation.

$*$	$f_*(\tau)$	$N_*$	$L(f_*, 2)$	$\alpha_*$
<b>A</b>	$\frac{\eta(4\tau)^5 \eta(8\tau)^5}{\eta(2\tau)^2 \eta(16\tau)^2}$	32	$\frac{\Gamma^2(\frac{1}{8})\Gamma^2(\frac{3}{8})}{64\sqrt{2}\pi}$	8
<b>B</b>	$\eta(4\tau)^6$	16	$\frac{\Gamma^4(\frac{1}{4})}{64\pi}$	8
<b>C</b>	$\eta(2\tau)^3 \eta(6\tau)^3$	12	$\frac{\Gamma^6(\frac{1}{3})}{2^{17/3}\pi^2}$	12
<b>D</b>	$\eta(4\tau)^6$	16	$\frac{\Gamma^4(\frac{1}{4})}{64\pi}$	16
<b>E</b>	$\eta(\tau)^2 \eta(2\tau) \eta(4\tau) \eta(8\tau)^2$	8	$\frac{\Gamma^2(\frac{1}{8})\Gamma^2(\frac{3}{8})}{192\pi}$	6
<b>F</b>	$q - 2q^2 + 3q^3 + \dots$	24	$\frac{\Gamma(\frac{1}{24})\Gamma(\frac{5}{24})\Gamma(\frac{7}{24})\Gamma(\frac{11}{24})}{96\sqrt{6}\pi}$	6

Table 2: The weight 3, level  $N_*$  newforms  $f_*$  with their  $L$ -values

Finally, we note that for sequence **E**, (9) can be written in terms of  $L(f_{\mathbf{E}}, 2)$  by virtue of the relation

$$L(f_{\mathbf{E}}, 1) = \frac{\sqrt{2}}{\pi} L(f_{\mathbf{E}}, 2). \quad (10)$$

This is an instance of a general principle, briefly discussed at the end of Section 3, which implies that the normalized critical  $L$ -values of  $f_{\mathbf{E}}(\tau)$  are algebraic multiples of each other.

## 2.2 Proof of Theorem 2.1

Zagier showed that each of the sporadic sequences  $C_*(n)$  in Table 1 has a modular parametrization, that is, there exists a modular function  $x(\tau)$  such that

$$y(\tau) := \sum_{n=0}^{\infty} C(n)x(\tau)^n \quad (11)$$

is a modular form of weight 1. In cases **C** and **E**, these are connected to the corresponding modular form  $f_*(\tau)$  in Table 2 in such a way that we can apply a general result of Verrill [36] to prove the modular congruences claimed in Theorem 2.1. This general result is an extension of Beukers' proof [9], which we revisit in Example 2.4, of the congruences (4) for the Apéry numbers.

**Theorem 2.3** ([36, Theorem 1.1]). *Let  $y(\tau)$  be a modular form of weight  $k$  and  $x(\tau)$  a modular function of level  $N$ , and define  $C(n)$  by (11). Suppose that, for some integers  $M$  and  $a_d$ ,*

$$y \frac{q}{x} \frac{dx}{dq} = \sum_{d|M} a_d f(d\tau),$$

where  $f(\tau) = \sum \gamma_n q^n$  is a weight  $k+2$ , level  $N$  Hecke eigenform with character  $\chi$ . Then,

$$C(mp^r) - \gamma_p C(mp^{r-1}) + \chi(p)p^{k+1}C(mp^{r-2}) \equiv 0 \pmod{p^r},$$

for any prime  $p \nmid NM$  and integers  $m, r$ . In particular, if  $C(1) = 1$ , then

$$C(p) \equiv \gamma_p \pmod{p}.$$

In the next example, we apply Theorem 2.3 to deduce the congruences (4) for the Apéry numbers  $A(n)$  (see also [36, Section 2.1]).

**Example 2.4.** As shown in [9], the Apéry numbers  $A(n)$  have the modular parametrization (11) with

$$x(\tau) = \frac{\eta(\tau)^{12}\eta(6\tau)^{12}}{\eta(2\tau)^{12}\eta(3\tau)^{12}}, \quad y(\tau) = \frac{\eta(2\tau)^7\eta(3\tau)^7}{\eta(\tau)^5\eta(6\tau)^5}.$$

Observe that, defining  $\tilde{x}(\tau)$  and  $\tilde{y}(\tau)$  by

$$\tilde{x}(\tau)^2 = x(2\tau), \quad \tilde{y}(\tau) = \tilde{x}(\tau)y(2\tau),$$

we have from (11) that

$$\tilde{y}(\tau) = \sum_{\substack{n=0 \\ n \text{ odd}}}^{\infty} A\left(\frac{n-1}{2}\right)\tilde{x}(\tau)^n.$$

It then follows from

$$\tilde{y} \frac{q}{\tilde{x}} \frac{d\tilde{x}}{dq} = f(\tau) - 9f(3\tau),$$

where  $f(\tau)$  is given by (3), and Theorem 2.3 that the congruences (4) hold for primes  $p > 3$ . This is the proof given in [9], which is generalized to Theorem 2.3 in [36].

We now proceed with the proof of Theorem 2.1.

*Proof of Theorem 2.1.* We first recall that the cases **C** and **D** were already proved in [35]. To alternatively deduce case **C** from Theorem 2.3, we note that  $C_{\mathbf{C}}(n)$  has the modular parametrization (11) with (see [38])

$$x(\tau) = \frac{\eta(\tau)^4\eta(6\tau)^8}{\eta(2\tau)^8\eta(3\tau)^4}, \quad y(\tau) = \frac{\eta(2\tau)^6\eta(3\tau)}{\eta(\tau)^3\eta(6\tau)^2}.$$

Defining  $\tilde{x}(\tau)$  and  $\tilde{y}(\tau)$  from  $x(\tau)$  and  $y(\tau)$  as in Example 2.4, it then follows from

$$\tilde{y} \frac{q}{\tilde{x}} \frac{d\tilde{x}}{dq} = f_{\mathbf{C}}(\tau)$$

and Theorem 2.3 that the congruences (7) hold for sequence **C**.

Similarly, it is shown in [38] that  $C_{\mathbf{E}}(n)$  has the modular parametrization (11) with

$$x(\tau) = \frac{\eta(\tau)^4 \eta(4\tau)^2 \eta(8\tau)^4}{\eta(2\tau)^{10}}, \quad y(\tau) = \frac{\eta(2\tau)^{10}}{\eta(\tau)^4 \eta(4\tau)^4}.$$

Again, defining  $\tilde{x}(\tau)$  and  $\tilde{y}(\tau)$  as in Example 2.4, it follows from

$$\tilde{y} \frac{q}{\tilde{x}} \frac{d\tilde{x}}{dq} = f_{\mathbf{E}}(\tau) + 2f_{\mathbf{E}}(2\tau)$$

and Theorem 2.3 that the congruences (7) hold for sequence  $\mathbf{E}$ . We now claim that

$$C_{\mathbf{A}}\left(\frac{p-1}{2}\right) \equiv \gamma_{p,\mathbf{A}} \pmod{p}, \quad (12)$$

where  $\gamma_{p,\mathbf{A}}$  is the  $p$ th Fourier coefficient of  $f_{\mathbf{A}}(\tau)$ . To see this, note that (see [35]) for primes  $p > 2$ ,

$$(-1)^{(p-1)/2} C_{\mathbf{A}}\left(\frac{p-1}{2}\right) \equiv \gamma_{p,\mathbf{E}} \pmod{p}, \quad (13)$$

(this congruence is recorded in [36, (4.55)] with the sign missing) where  $\gamma_{p,\mathbf{E}}$  is the  $p$ th Fourier coefficient of  $f_{\mathbf{E}}(\tau)$ . Now, observe the relation

$$(-1)^{(n-1)/2} \gamma_{n,\mathbf{A}} = \gamma_{n,\mathbf{E}} + 2\gamma_{n/2,\mathbf{E}}$$

for all integers  $n \geq 1$ . In particular, for odd  $n$ ,  $\gamma_{n,\mathbf{E}} = (-1)^{(n-1)/2} \gamma_{n,\mathbf{A}}$ . Thus, (13) is equivalent to (12). We note that (13) provides a quick alternative proof of the congruences for sequence  $\mathbf{E}$  by showing that

$$C_{\mathbf{E}}\left(\frac{p-1}{2}\right) \equiv (-1)^{(p-1)/2} C_{\mathbf{A}}\left(\frac{p-1}{2}\right) \pmod{p}.$$

This congruence can be deduced directly from the binomial sums recorded in Table 1 and the fact that the congruences hold termwise.

Finally, let us prove the congruences for sequence  $\mathbf{B}$ . Expressing the defining binomial sum hypergeometrically, we have

$$C_{\mathbf{B}}\left(\frac{p-1}{2}\right) = 3^{(p-1)/2} {}_3F_2 \left[ \begin{matrix} \frac{1-p}{6}, \frac{3-p}{6}, \frac{5-p}{6} \\ 1, 1 \end{matrix} \middle| 1 \right].$$

Because the hypergeometric series is a finite sum (one of the top parameters is a negative integer), it follows that

$$C_{\mathbf{B}}\left(\frac{p-1}{2}\right) \equiv 3^{(p-1)/2} {}_3F_2 \left[ \begin{matrix} \frac{1-p}{6}, \frac{3-p}{6}, \frac{5-p}{6} \\ 1 - \frac{p}{6}, 1 - \frac{p}{3} \end{matrix} \middle| 1 \right] \pmod{p}.$$

By specializing Watson's identity (see, for instance, [4, Theorem 3.5.5(i)]), we find that this hypergeometric sum has the closed form evaluation

$${}_3F_2 \left[ \begin{matrix} \frac{1-p}{6}, \frac{3-p}{6}, \frac{5-p}{6} \\ 1 - \frac{p}{6}, 1 - \frac{p}{3} \end{matrix} \middle| 1 \right] = \left( \frac{\Gamma(\frac{1}{2})\Gamma(1 - \frac{p}{6})}{\Gamma(\frac{7-p}{12})\Gamma(\frac{11-p}{12})} \right)^2. \quad (14)$$

If  $p \equiv 3 \pmod{4}$ , then  $p \equiv 7, 11 \pmod{12}$  and we see that the right-hand side of (14) is zero, so that  $C_{\mathbf{B}}(\frac{p-1}{2})$  vanishes modulo  $p$ . Suppose that  $p \equiv 1 \pmod{4}$ . With some care, we are able to write

$$\frac{\Gamma(\frac{1}{2})\Gamma(1-\frac{p}{6})}{\Gamma(\frac{7-p}{12})\Gamma(\frac{11-p}{12})} \equiv \frac{\Gamma_p(\frac{1}{2})\Gamma_p(1-\frac{p}{6})}{\Gamma_p(\frac{7-p}{12})\Gamma_p(\frac{11-p}{12})} \equiv -\frac{\Gamma_p(\frac{1}{2})}{\Gamma_p(\frac{7}{12})\Gamma_p(\frac{11}{12})} \pmod{p},$$

where  $\Gamma_p$  is Morita's  $p$ -adic gamma function (see, for instance, [15, 11.6] or [21, IV.2]). Since  $\Gamma_p(1/2)^2 = (-1)^{(p+1)/2}$ , it follows that, if  $p \equiv 1 \pmod{4}$ , then

$$C_{\mathbf{B}}(\frac{p-1}{2}) \equiv -\frac{3^{(p-1)/2}}{\Gamma_p(\frac{7}{12})^2\Gamma_p(\frac{11}{12})^2} \equiv -3^{(p-1)/2}\Gamma_p(\frac{1}{12})^2\Gamma_p(\frac{5}{12})^2 \pmod{p},$$

where we used the  $p$ -adic version of the reflection formula for the final congruence. On the other hand, it follows from the  $p$ -adic Gauss–Legendre multiplication formula (see, for instance, [15, 11.6.14] or [21, p. 91]) that, for primes  $p \equiv 1 \pmod{4}$ ,

$$\Gamma_p(\frac{1}{12})^2\Gamma_p(\frac{5}{12})^2 = (\frac{3}{p})\Gamma_p(\frac{1}{4})^4.$$

Since  $3^{(p-1)/2} \equiv (\frac{3}{p}) \pmod{p}$ , we conclude that, modulo  $p$ ,

$$C_{\mathbf{B}}(\frac{p-1}{2}) \equiv \begin{cases} -\Gamma_p(\frac{1}{4})^4, & \text{if } p \equiv 1 \pmod{4}, \\ 0, & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Comparing with the congruences established for sequence  $\mathbf{D}$  in [35, (13.4)], we arrive at

$$C_{\mathbf{B}}(\frac{p-1}{2}) \equiv C_{\mathbf{D}}(\frac{p-1}{2}) \pmod{p}, \quad (15)$$

which implies the claimed congruences for sequence  $\mathbf{B}$ .  $\square$

### 2.3 Proof of Theorem 2.2

This section is devoted to proving Theorem 2.2. We first prove case  $\mathbf{D}$  in detail, then briefly indicate how to establish cases  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$  and  $\mathbf{E}$ . We conclude with a sketch of case  $\mathbf{F}$ . We note that the relation of the hypergeometric series, which arise for sequences  $\mathbf{A}$ ,  $\mathbf{C}$ ,  $\mathbf{D}$ , and the corresponding  $L$ -values already appears in [40].

*Proof of Theorem 2.2.* We first claim that

$$C_{\mathbf{D}}(-\frac{1}{2}) = \frac{16}{\pi^2}L(f_{\mathbf{D}}, 2). \quad (16)$$

Expressing the defining binomial sum hypergeometrically, we have

$$C_{\mathbf{D}}(x) = {}_3F_2 \left[ \begin{matrix} -x, -x, x+1 \\ 1, 1 \end{matrix} \middle| 1 \right],$$



so that, in particular,

$$C_{\mathcal{D}}(-\tfrac{1}{2}) = {}_3F_2 \left[ \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} \middle| 1 \right].$$

We could evaluate the right-hand side using hypergeometric identities (such as, in this case, [4, Theorem 3.5.5]). Instead, here and in subsequent cases, we find it more fitting to the overall theme to employ modular parametrizations. As such, applying Clausen's identity (see, for instance, [12, Proposition 5.6])

$${}_3F_2 \left[ \begin{matrix} \frac{1}{2}, s, 1-s \\ 1, 1 \end{matrix} \middle| 4x(1-x) \right] = {}_2F_1 \left[ \begin{matrix} s, 1-s \\ 1 \end{matrix} \middle| x \right]^2, \quad (17)$$

with  $s = 1/2$  and the modular parametrization (see [11], [37, p.63] or [39, (37)])

$${}_2F_1 \left[ \begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \middle| \lambda(\tau) \right] = \theta_3(\tau)^2, \quad (18)$$

where  $\theta_2(\tau) = \sum_{n \in \mathbb{Z} + 1/2} q^{n^2/2}$ ,  $\theta_3(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2/2}$  and  $\lambda(\tau) = \left( \frac{\theta_2(\tau)}{\theta_3(\tau)} \right)^4$ , we find that

$$C_{\mathcal{D}}(-\tfrac{1}{2}) = {}_2F_1 \left[ \begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \middle| \frac{1}{2} \right]^2 = \left( \frac{\Gamma^2(\frac{1}{4})}{2\pi^{3/2}} \right)^2, \quad (19)$$

upon taking  $\tau = i$ , in which case  $\lambda(i) = \frac{1}{2}$  and  $\theta_3(i)^2 = \frac{\Gamma^2(1/4)}{2\pi^{3/2}}$ . On the other hand, it is shown by Rogers, Wan and Zucker [30] that

$$L(f_{\mathcal{D}}, 2) = \frac{\Gamma^4(\frac{1}{4})}{64\pi}.$$

In light of (19), this proves (16). Next, we claim that

$$C_{\mathcal{A}}(-\tfrac{1}{2}) = \frac{8}{\pi^2} L(f_{\mathcal{A}}, 2). \quad (20)$$

Proceeding as above, we find that

$$C_{\mathcal{A}}(-\tfrac{1}{2}) = {}_3F_2 \left[ \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} \middle| -1 \right].$$

Employing (17) and the modular parametrization, we obtain

$$C_{\mathcal{A}}(-\tfrac{1}{2}) = {}_2F_1 \left[ \begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \middle| \frac{1-\sqrt{2}}{2} \right]^2 = \theta_3 \left( 1 + i\sqrt{2} \right)^4 = \frac{\Gamma^2(\frac{1}{8})\Gamma^2(\frac{3}{8})}{8\sqrt{2}\pi^3}. \quad (21)$$

Again, up to the factor  $8/\pi^2$ , this matches the corresponding  $L$ -value evaluation [30]

$$L(f_{\mathcal{A}}, 2) = \frac{\Gamma^2(\frac{1}{8})\Gamma^2(\frac{3}{8})}{64\sqrt{2}\pi}.$$

This proves (20). Now, in order to see

$$C_{\mathbf{B}}(-\tfrac{1}{2}) = \frac{8}{\pi^2} L(f_{\mathbf{B}}, 2), \quad (22)$$

we begin with

$$C_{\mathbf{B}}(x) = 3^x {}_3F_2 \left[ \begin{matrix} -\frac{x}{3}, -\frac{x-1}{3}, -\frac{x-2}{3} \\ 1, 1 \end{matrix} \middle| 1 \right]$$

and hence

$$C_{\mathbf{B}}(-\tfrac{1}{2}) = 3^{-1/2} {}_3F_2 \left[ \begin{matrix} \frac{1}{6}, \frac{1}{2}, \frac{5}{6} \\ 1, 1 \end{matrix} \middle| 1 \right].$$

Let  $j(\tau)$  denote Klein's modular function. By [12, Theorem 5.7], we have

$${}_3F_2 \left[ \begin{matrix} \frac{1}{6}, \frac{1}{2}, \frac{5}{6} \\ 1, 1 \end{matrix} \middle| \frac{1728}{j(\tau)} \right] = \sqrt{1 - \lambda(\tau)(1 - \lambda(\tau))} {}_2F_1 \left[ \begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \middle| \lambda(\tau) \right]^2,$$

which specialized to  $\tau = i$ , and combined with (18), yields

$$C_{\mathbf{B}}(-\tfrac{1}{2}) = 3^{-1/2} {}_3F_2 \left[ \begin{matrix} \frac{1}{6}, \frac{1}{2}, \frac{5}{6} \\ 1, 1 \end{matrix} \middle| 1 \right] = \frac{1}{2} \theta_3(i)^4 = \frac{\Gamma^4(\frac{1}{4})}{8\pi^3}.$$

Up to the factor  $8/\pi^2$ , this equals the  $L$ -value evaluation [30]

$$L(f_{\mathbf{B}}, 2) = \frac{\Gamma^4(\frac{1}{4})}{64\pi}.$$

Thus, (22) follows. To prove

$$C_{\mathbf{C}}(-\tfrac{1}{2}) = \frac{12}{\pi^2} L(f_{\mathbf{C}}, 2), \quad (23)$$

we first observe

$$C_{\mathbf{C}}(-\tfrac{1}{2}) = \operatorname{Re} {}_3F_2 \left[ \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} \middle| 4 \right].$$

Employing (17) and the modular parametrization, we obtain

$${}_3F_2 \left[ \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} \middle| 4 \right] = {}_2F_1 \left[ \begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \middle| \frac{1-i\sqrt{3}}{2} \right]^2 = \theta_3 \left( -\frac{1-i\sqrt{3}}{2} \right)^4 = \frac{(3-i\sqrt{3}) \Gamma^6(\frac{1}{3})}{2^{11/3} \pi^4}.$$

Up to the factor  $12/\pi^2$ , the real part of this equals the  $L$ -value evaluation [30]

$$L(f_{\mathbf{C}}, 2) = \frac{\Gamma^6(\frac{1}{3})}{2^{17/3} \pi^2}.$$

This yields (23). Next, to deduce

$$\operatorname{res}_{x=-1/2} C_{\mathbf{E}}(x) = \frac{6}{\pi^2} L(f_{\mathbf{E}}, 1), \quad (24)$$

we start with

$$C_{\mathbf{E}}(x) = \binom{2x}{x} {}_3F_2 \left[ \begin{matrix} -x, -x, \frac{1}{2} \\ \frac{1}{2} - x, 1 \end{matrix} \middle| -1 \right]$$

and hence

$$\operatorname{res}_{x=-1/2} C_{\mathbf{E}}(x) = \frac{1}{2\pi} {}_3F_2 \left[ \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} \middle| -1 \right] = \frac{\Gamma^2(\frac{1}{8})\Gamma^2(\frac{3}{8})}{16\sqrt{2}\pi^4},$$

where the second equality is a consequence of (21). Up to the factor  $6/\pi^2$ , this equals

$$L(f_{\mathbf{E}}, 1) = \frac{\Gamma^2(\frac{1}{8})\Gamma^2(\frac{3}{8})}{96\sqrt{2}\pi^2},$$

which follows from (10) and the value for  $L(f_{\mathbf{E}}, 2)$  obtained in [30]. This proves (24). Finally, we claim that

$$C_{\mathbf{F}}(-\frac{1}{2}) = \frac{6}{\pi^2} L(f_{\mathbf{F}}, 2) = \frac{\Gamma(\frac{1}{24})\Gamma(\frac{5}{24})\Gamma(\frac{7}{24})\Gamma(\frac{11}{24})}{16\sqrt{6}\pi^3}. \quad (25)$$

To begin with, note that

$$C_{\mathbf{F}}(-\frac{1}{2}) = \frac{1}{\sqrt{8}} \sum_{k=0}^{\infty} 2^{-5k} \binom{2k}{k} C_{\mathbf{A}}(k) = \frac{1}{\sqrt{8}} g\left(\frac{1}{32}\right),$$

where

$$g(z) = \sum_{k=0}^{\infty} z^k \binom{2k}{k} \sum_{j=0}^k \binom{k}{j}^3.$$

In [14, Theorem 2.1], the modular parametrization

$$g\left(\frac{x(\tau)}{(1-x(\tau))^2}\right) = \frac{1}{6}(6E_2(6\tau) + 3E_2(3\tau) - 2E_2(2\tau) - E_2(\tau)),$$

with

$$x(\tau) = \left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)}\right)^{12}, \quad E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n},$$

is obtained. Specializing this parametrization at  $\tau = \tau_0 = i/\sqrt{6}$ , we obtain the desired value  $g(1/32)$ . It then is a standard application of the Chowla–Selberg formula [32] to show that

$$3E_2(3\tau_0) - E_2(\tau_0) = 6E_2(6\tau_0) - 2E_2(2\tau_0) = \frac{\sqrt{3}\Gamma(\frac{1}{24})\Gamma(\frac{5}{24})\Gamma(\frac{7}{24})\Gamma(\frac{11}{24})}{8\pi^3},$$

which implies

$$C_{\mathbf{F}}(-\frac{1}{2}) = \frac{\Gamma(\frac{1}{24})\Gamma(\frac{5}{24})\Gamma(\frac{7}{24})\Gamma(\frac{11}{24})}{16\sqrt{6}\pi^3}. \quad (26)$$

That the right-hand side equals the claimed  $L$ -value then follows from work of Damerell [17] because  $L(f_{\mathbf{F}}, s)$  can be viewed as a Hecke  $L$ -series on the field  $\mathbb{Q}(\sqrt{-6})$  (see also [7], [31]).  $\square$

## 2.4 Interpolating the sporadic sequences

Zagier's interpolated series (1) is absolutely convergent for all  $x \in \mathbb{C}$  (as a consequence of (30)) and defines a holomorphic function satisfying the symmetry  $A(x) = A(-x - 1)$ . Zagier shows the somewhat surprising fact that  $A(x)$  does not satisfy the same recurrence as the Apéry numbers, but instead the inhomogeneous functional equation

$$P(x, S_x)A(x) = \frac{8}{\pi^2}(2x + 3)\sin^2(\pi x) \quad (27)$$

for all complex  $x$ , where

$$P(x, S_x) = (x + 2)^3 S_x^2 - (2x + 3)(17x^2 + 51x + 39)S_x + (x + 1)^3 \quad (28)$$

is Apéry's recurrence operator, and  $S_x$  denotes the (forward) shift operator in  $x$ , meaning that  $S_x f(x) = f(x + 1)$ .

**Remark 2.5.** Let us illustrate how one can algorithmically derive and prove (27). Let  $D(x, k)$  be the summand in the sum defining  $A(x)$ . Creative telescoping applied to  $D(x, k)$  determines the operator  $P(x, S_x)$  given in (28) as well as a rational function  $R(x, k)$  such that

$$P(x, S_x)D(x, k) = (1 - S_k)R(x, k)D(x, k). \quad (29)$$

It follows that

$$P(x, S_x) \sum_{k=0}^{K-1} D(x, k) = R(x, 0)D(x, 0) - R(x, K)D(x, K) = -R(x, K)D(x, K),$$

and it remains to compute the limit of the right-hand side as  $K \rightarrow \infty$ . Using basic properties of the gamma function, as done in [39], one obtains

$$D(x, k) = \left[ \frac{\sin(\pi x)}{\pi k} \right]^2 + O\left(\frac{1}{k^3}\right), \quad k \rightarrow \infty, \quad (30)$$

from which we deduce that  $-R(x, K)D(x, K)$  approaches  $8(2x + 3)\sin^2(\pi x)/\pi^2$  as  $K \rightarrow \infty$ . The following lines of Mathematica code use Koutschan's Mathematica package `HolonomicFunctions` [23] to perform all of these computations automatically:

```
Dxk = Binomial[x,k]^2 Binomial[x+k,k]^2
{{P}, {R}} = CreativeTelescoping[Dxk, S[k]-1, {S[x]}]
{R} = OrePolynomialListCoefficients[R]
Limit[-R Dxk, k->Infinity, Assumptions->Element[k,Integers]]
```

**Remark 2.6.** We note that the sum in (1) actually has natural boundaries, meaning that the range of summation can be extended from nonnegative integers

to all integers without changing the sum. The reason is that the summand vanishes for all  $x \in \mathbb{C}$  if  $k$  is a negative integer. More specifically, if  $k$  is a negative integer, then  $\binom{x+k}{k} = 0$  for all  $x \in \mathbb{C} \setminus \{-k-1, -k-2, \dots, 1, 0\}$ , while  $\binom{x}{k} = 0$  for all  $x \in \mathbb{C} \setminus \{-1, -2, \dots, k\}$ . For more details on binomial coefficients with negative integer entries, we refer to [18] and [26].

Somewhat unexpectedly, there are marked differences when considering the interpolations of the sporadic sequences given in Table 1. For illustration, consider sequence  $\mathbf{D}$  with interpolation

$$C_{\mathbf{D}}(x) = \sum_{k=0}^{\infty} \binom{x}{k}^2 \binom{x+k}{k}. \quad (31)$$

In this case, we find that, as  $k \rightarrow \infty$ ,

$$\binom{x}{k}^2 \binom{x+k}{k} \sim \frac{\Gamma(x+1)}{k^x} \left[ \frac{\sin(\pi x)}{\pi k} \right]^2,$$

which implies that the series (31) converges if  $\operatorname{Re} x > -1$  but diverges if  $\operatorname{Re} x < -1$ . Moreover, proceeding as in the case of the Apéry numbers  $A(n)$ , it follows that  $C_{\mathbf{D}}(x)$  satisfies the homogeneous functional equation

$$[(x+2)^2 S_x^2 - (11x^2 + 33x + 25)S_x - (x+1)^2]C_{\mathbf{D}}(x) = 0$$

for all complex  $x$  with  $\operatorname{Re} x > -1$ . This is recurrence (5) with  $(a, b, c) = (11, 3, -1)$ .

The situation is similar for our interpolations of the sequences  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{E}$ . In each case, the defining series (see Table 1) converges if  $\operatorname{Re} x > -1$  and one finds, as in the case of sequence  $\mathbf{D}$ , that the interpolation satisfies the recurrence (5) for the appropriate choice of  $(a, b, c)$ .

**Example 2.7.** Some care is required for sequence  $\mathbf{C}$ , which has the binomial sum representation

$$C_{\mathbf{C}}(n) = \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k}.$$

In this case, letting  $n$  be a complex variable and extending the summation over all nonnegative integers  $k$  never yields a convergent sum (unless  $n$  is a nonnegative integer, in which case the sum is a finite one). However, the binomial sum can be expressed hypergeometrically as

$$C_{\mathbf{C}}(n) = {}_3F_2 \left[ \begin{matrix} -n, -n, \frac{1}{2} \\ 1, 1 \end{matrix} \middle| 4 \right]. \quad (32)$$

For integers  $n \geq 0$ , this hypergeometric series is a finite sum. For other values of  $n$ , we can make sense of the hypergeometric function (32) by replacing 4 with a complex argument  $z$  (the series converges for  $|z| < 1$ ) and analytic continuation to  $z = 4$ . As usual, the principal branch of the hypergeometric function is

chosen by cutting from  $z = 1$  to  $z = \infty$  on the real axis. As a consequence, there is a choice to approach  $z = 4$  from either above or below the real axis, and the two resulting values are complex conjugates of each other. We avoid this ambiguity, as well as complex values, by defining

$$C_C(x) = \operatorname{Re} {}_3F_2 \left[ \begin{matrix} -x, -x, \frac{1}{2} \\ 1, 1 \end{matrix} \middle| 4 \right].$$

That this is a sensible choice of interpolation is supported by Theorem 2.2.

**Example 2.8.** For sequence  $\mathbf{F}$ , let us consider the interpolation

$$C_{\mathbf{F}}(x) = \sum_{k=0}^{\infty} (-1)^k 8^{x-k} \binom{x}{k} C_{\mathbf{A}}(k), \quad (33)$$

where  $C_{\mathbf{A}}(n)$  are the Franel numbers

$$C_{\mathbf{A}}(n) = \sum_{k=0}^n \binom{n}{k}^3 = \frac{2\sqrt{3}}{\pi} \frac{2^{3n}}{3n} \left( 1 + O\left(\frac{1}{n}\right) \right).$$

It follows that, as  $k \rightarrow \infty$ ,

$$(-1)^k 8^{x-k} \binom{x}{k} C_{\mathbf{A}}(k) = \frac{2}{\pi\sqrt{3}} \frac{8^x}{\Gamma(-x)} \frac{1}{k^{x+2}} \left( 1 + O\left(\frac{1}{k}\right) \right),$$

from which we deduce that, once more, the series (33) converges if  $\operatorname{Re} x > -1$ . Consequently, we expect that the truncation

$$C_{\mathbf{F}}(x; N) = \sum_{k=0}^N (-1)^k 8^{x-k} \binom{x}{k} C_{\mathbf{A}}(k),$$

as  $N \rightarrow \infty$ , has an asymptotic expansion of the form

$$C_{\mathbf{F}}(x; N) = C_{\mathbf{F}}(x) + \frac{b_1(x)}{N^{x+1}} + \frac{b_2(x)}{N^{x+2}} + \dots$$

Using this assumption, we can speed up the convergence of  $C_{\mathbf{F}}(-1/2; N)$  by considering the sequence  $c_n = C_{\mathbf{F}}(-1/2; n^2)$  and approximating its limit via the differences  $(S_n - 1)^m n^m c_n / m!$  for suitable choices of  $m$  and  $n$ . This allows us to compute  $C_{\mathbf{F}}(-1/2)$  to, say, 50 decimal places. Namely,

$$C_{\mathbf{F}}(-\tfrac{1}{2}) = 0.50546201971732600605200405322714025998512901481742\dots$$

This allowed us to numerically discover (8) for sequence  $\mathbf{F}$ . For comparison, summing the first 100,000 terms of the series only produces three correct digits.

### 3 Cellular integrals

Recently, Brown [13] introduced a program where period integrals on the moduli space  $\mathcal{M}_{0,N}$  of curves of genus 0 with  $N$  marked points play a central role in understanding irrationality proofs of values of the Riemann zeta function. The idea is to associate a rational function  $f_\sigma$  and a differential  $(N-3)$ -form  $\omega_\sigma$  to a given permutation  $\sigma = \sigma_N$  on  $\{1, 2, \dots, N\}$ . Consider the cellular integral

$$I_\sigma(n) := \int_{S_N} f_\sigma^n \omega_\sigma,$$

where

$$S_N = \{(t_1, \dots, t_{N-3}) \in \mathbb{R}^{N-3} : 0 < t_1 < \dots < t_{N-3} < 1\}.$$

By [13, Corollary 8.2],  $I_\sigma(n)$  is a  $\mathbb{Q}$ -linear combination of multiple zeta values of weight less than or equal to  $N-3$ . Suppose that this linear combination is of the form  $A_{\sigma_N}(n)\zeta_\sigma(N-3)$ , with  $A_{\sigma_N}(n) \in \mathbb{Q}$ , plus a combination of multiple zeta values of weight less than  $N-3$ . We then say that  $A_\sigma(n) = A_{\sigma_N}(n)$  is the *leading coefficient* of the cellular integral  $I_\sigma(n)$ . For example, if  $N=5$ , then  $\sigma_5 = (1, 3, 5, 2, 4)$  is the unique convergent permutation,  $I_{\sigma_5}(n)$  recovers Beukers' integral for  $\zeta(2)$  [8] and the leading coefficients  $A_{\sigma_5}(n)$  are the Apéry numbers  $C_{\mathcal{D}}(n)$  in (6).

In [27], an explicit family  $\sigma_N$  of convergent configurations for odd  $N \geq 5$  is constructed such that the leading coefficients  $A_{\sigma_N}(n)$  are powers of the Apéry numbers  $C_{\mathcal{D}}(n)$ , that is,

$$A_{\sigma_N}(n) = C_{\mathcal{D}}(n)^{(N-3)/2}. \quad (34)$$

The first main result in [27] extends Theorem 2.1 for sequence  $\mathcal{D}$  to a supercongruence for all odd weights greater than or equal to 3. Specifically, for odd  $k = N-2 \geq 3$ , consider the binary theta series

$$f_k(\tau) = \frac{1}{4} \sum_{(n,m) \in \mathbb{Z}^2} (-1)^{m(k-1)/2} (n-im)^{k-1} q^{n^2+m^2} =: \sum_{n \geq 1} \gamma_k(n) q^n. \quad (35)$$

**Theorem 3.1** ([27, Theorem 1.1]). *For each odd integer  $N \geq 5$ , let  $A_{\sigma_N}(n)$  and  $f_k(\tau)$  be as in (34) and (35), respectively. Then, for all primes  $p \geq 5$ ,*

$$A_{\sigma_N}\left(\frac{p-1}{2}\right) \equiv \gamma_k(p) \pmod{p^2}. \quad (36)$$

Using the interpolation (31) for  $C_{\mathcal{D}}(n)$  and (34), we have the following analogue of (2) for all odd  $N \geq 5$ .

**Theorem 3.2.** *Let  $N \geq 5$  be an odd positive integer,  $k = N-2$  and  $f_k(\tau)$  be as in (35). Then,*

$$A_{\sigma_N}\left(-\frac{1}{2}\right) = \frac{\alpha_k}{\pi^{k-1}} L(f_k, k-1), \quad (37)$$

where  $\alpha_k$  is an explicit rational number given as follows:

$$\alpha_k = 2^{(k+1)/2} (k-2) \begin{cases} 2/r_{(k-1)/2}, & \text{if } k \equiv 1 \pmod{4}, \\ 1/s_{(k-1)/2}, & \text{if } k \equiv 3 \pmod{4}. \end{cases} \quad (38)$$

Here,  $r_n$  is defined by  $r_2 = 1/5$ ,  $r_3 = 0$  and

$$(2n+1)(n-3)r_n = 3 \sum_{k=2}^{n-2} r_k r_{n-k} \quad (39)$$

for  $n \geq 4$ , and  $s_n$  is defined by  $s_1 = 1/4$ ,  $s_2 = 11/80$ ,  $s_3 = 1/32$  and the same recursion (39) for  $n \geq 4$ .

*Proof of Theorem 3.2.* Since  $f_3(\tau) = \eta(4\tau)^6$ , the case  $N = 5$  is (16). Thus, we assume  $N > 5$ . As a consequence of (19) and (34), we have

$$A_{\sigma_N}(-\frac{1}{2}) = \left( \frac{\Gamma^2(\frac{1}{4})}{2\pi^{3/2}} \right)^{N-3} = \left( \frac{\sqrt{2}\omega}{\pi} \right)^{k-1}, \quad (40)$$

where

$$\omega = 2 \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\Gamma^2(\frac{1}{4})}{2\sqrt{2}\pi}$$

is the lemniscate constant. On the other hand, it follows from the representation (35) that

$$\begin{aligned} L(f_k, k-1) &= \frac{1}{4} \sum_{(n,m) \neq (0,0)} (-1)^{m(k-1)/2} \frac{(n-im)^{k-1}}{(n^2+m^2)^{k-1}} \\ &= \frac{1}{4} \sum_{(n,m) \neq (0,0)} (-1)^{m(k-1)/2} \frac{1}{(n+im)^{k-1}}. \end{aligned}$$

In other words, these  $L$ -values are values of the Eisenstein series

$$G_\ell(\tau) = \sum_{(n,m) \neq (0,0)} \frac{1}{(n+m\tau)^\ell}$$

of even weight  $\ell$ . Specifically, since

$$2G_\ell(2\tau) - G_\ell(\tau) = \sum_{(n,m) \neq (0,0)} \frac{(-1)^m}{(n+m\tau)^\ell},$$

we have

$$L(f_k, k-1) = \frac{1}{4} \begin{cases} G_{k-1}(i), & \text{if } k \equiv 1 \pmod{4}, \\ 2G_{k-1}(2i) - G_{k-1}(i), & \text{if } k \equiv 3 \pmod{4}. \end{cases}$$

We note that, if  $k \equiv 3 \pmod{4}$ , then  $G_{k-1}(i) = 0$  because, writing  $k = 4\ell + 3$ ,

$$\begin{aligned} G_{k-1}(i) &= \sum_{(n,m) \neq (0,0)} \frac{1}{(n+mi)^{4\ell+2}} \\ &= \sum_{(n,m) \neq (0,0)} \frac{1}{i^{4\ell+2}(m-ni)^{4\ell+2}} = -G_{k-1}(i). \end{aligned}$$



For  $n \geq 4$ , we have (see [6, Theorem 1.13])

$$(4n^2 - 1)(n - 3)G_{2n} = 3 \sum_{k=2}^{n-2} (2k - 1)(2n - 2k - 1)G_{2k}G_{2(n-k)},$$

which, upon setting  $H_k = (2k - 1)G_{2k}$ , takes the simplified form

$$(2n + 1)(n - 3)H_n = 3 \sum_{k=2}^{n-2} H_k H_{n-k}. \quad (41)$$

In terms of the functions  $H_k$ , we have

$$L(f_k, k - 1) = \frac{1}{4(k - 2)} \begin{cases} H_{2\ell}(i), & \text{if } k = 4\ell + 1, \\ 2H_{2\ell+1}(2i), & \text{if } k = 4\ell + 3. \end{cases}$$

Note that the required values of  $H_k(\tau)$  at  $\tau = i$  and  $\tau = 2i$  are determined by the recursive relation (41) once we know the initial cases  $k = 2$  and  $k = 3$ . It is shown, for instance, in [24, Theorem 6] that

$$H_2(i) = 3G_4(i) = \frac{\omega^4}{5},$$

and our earlier discussion implies  $H_3(i) = 5G_6(i) = 0$ . Similarly, one shows that

$$H_2(2i) = 3G_4(2i) = \frac{11\omega^4}{80}, \quad H_3(2i) = 5G_6(2i) = \frac{\omega^6}{32}.$$

In light of these initial values, the recurrence (41) implies that, for  $n \geq 2$ , the values  $H_n(i)$  and  $H_n(2i)$  are rational multiples of  $\omega^{2n}$ . Moreover, the rational factors are given by the sequences  $r_n$  and  $s_n$ :

$$r_n = \frac{H_n(i)}{\omega^{2n}}, \quad s_n = \frac{H_n(2i)}{\omega^{2n}}.$$

Thus,

$$L(f_k, k - 1) = \frac{\omega^{k-1}}{4(k - 2)} \begin{cases} r_{2\ell}, & \text{if } k = 4\ell + 1, \\ 2s_{2\ell+1}, & \text{if } k = 4\ell + 3, \end{cases}$$

and the claim then follows from comparison with (40).  $\square$

**Remark 3.3.** Let us indicate that the rational numbers featuring in Theorem 3.2 are arithmetically interesting in their own right, and analogous to Bernoulli numbers. The values  $G_{4\ell}(i)$  were first explicitly evaluated by Hurwitz [20] (see [24] for a modern account), who showed that

$$G_{4\ell}(i) = \sum_{(n,m) \neq (0,0)} \frac{1}{(n + im)^{4\ell}} = \frac{(2\omega)^{4\ell}}{(4\ell)!} E_\ell, \quad (42)$$

where the  $E_\ell$  are positive rational numbers characterized by  $E_1 = 1/10$  and the recurrence

$$E_n = \frac{3}{(2n-3)(16n^2-1)} \sum_{k=1}^{n-1} (4k-1)(4n-4k-1) \binom{4n}{4k} E_k E_{n-k}.$$

In terms of the numbers  $r_n$  defined in Theorem 3.2, we have

$$E_n = \frac{(4n)!}{2^{4n}} \frac{r_{2n}}{4n-1}.$$

Equation (42), defining the Hurwitz numbers  $E_\ell$ , can be seen as an analog of

$$\sum_{n \neq 0} \frac{1}{n^{2\ell}} = \frac{(2\pi)^{2\ell}}{(2\ell)!} B_\ell$$

characterizing the Bernoulli numbers  $B_\ell$ . In other words, in the theory of Gaussian integers the Hurwitz numbers  $E_\ell$  play a role comparable to that played by the Bernoulli numbers for the usual integers. That this analogy extends much further, including to the theorem of von Staudt–Clausen, is beautifully demonstrated by Hurwitz [20].

**Example 3.4.** Let us make the case  $N = 7$  of Theorem 3.2 explicit. The leading coefficients  $A_{\sigma_7}(n)$  are the squares of the Apéry numbers  $C_{\mathcal{D}}(n)$  and the modular form  $f_5(\tau)$  can alternatively be expressed as

$$f_5(\tau) = \eta(\tau)^4 \eta(2\tau)^2 \eta(4\tau)^4.$$

The Zagier-type  $L$ -value evaluation proven in Theorem 3.2 is

$$A_{\sigma_7}(-\tfrac{1}{2}) = \frac{240}{\pi^2} L(f_5, 4).$$

It is observed in [30] that this and many other  $L$ -values are naturally expressed in terms of integrals of the complete elliptic integral  $K$ ; for instance,

$$L(f_5, 4) = \frac{1}{30} \int_0^1 K'(k)^3 dk = \frac{1}{9} \int_0^1 K(k)^3 dk.$$

**Example 3.5.** The values of the first several  $\alpha_k$  in Theorem 3.2 are  $\alpha_3 = 16$ ,  $\alpha_5 = 240$ ,  $\alpha_7 = 2560$ ,  $\alpha_9 = 33600$ ,  $\alpha_{11} = 491520$ ,  $\alpha_{13} = 6864000$  and  $\alpha_{15} = \frac{1022361600}{11}$ .

Let  $L^*(f, s) = (2\pi)^{-s} \Gamma(s) L(f, s)$  be the normalized  $L$ -function for  $f$ . It follows from the work of Eichler, Shimura and Manin on period polynomials (for example, see [34]) that the critical  $L$ -values  $L^*(f, s)$  for odd  $s$  (as well as those for even  $s$ ) are algebraic multiples of each other. Moreover, if  $f$  has odd weight  $k$ , then by virtue of the functional equation all critical  $L$ -values  $L^*(f, s)$

are algebraic multiples of each other. In particular, it follows that (37) can be rewritten as

$$A_{\sigma_N}(-\frac{1}{2}) = \beta_k \frac{L(f_k, 2)}{\pi^2}$$

for some algebraic numbers  $\beta_k$ . In fact, it appears that the  $\beta_k$ 's are rational numbers.

**Example 3.6.** Numerically, the first several values of  $\beta_k$  are  $\beta_3 = 16$ ,  $\beta_5 = 48$ ,  $\beta_7 = 4$ ,  $\beta_9 = 14$ ,  $\beta_{11} = \frac{1}{33}$ ,  $\beta_{13} = \frac{11}{18}$ ,  $\beta_{15} = \frac{1}{33156}$ . These values, as well as the relations indicated in Example 3.7, may in principle be rigorously obtained using, for instance, Rankin's method [33].

**Example 3.7.** As indicated above, all critical  $L$ -values  $L^*(f_k, s)$  are algebraic multiples of each other. In fact, numerical computations suggest that all critical  $L$ -values are rationally related. The first few cases are:

$$\begin{aligned} L(f_5, 4) &= \frac{2\pi}{5}L(f_5, 3) = \frac{\pi^2}{5}L(f_5, 2) = \frac{\pi^3}{6}L(f_5, 1), \\ L(f_7, 6) &= \frac{3\pi}{10}L(f_7, 5) = \frac{3\pi^2}{40}L(f_7, 4) = \frac{\pi^3}{80}L(f_7, 3) = \frac{\pi^4}{640}L(f_7, 2) \\ &= \frac{\pi^5}{3840}L(f_7, 1), \\ L(f_9, 8) &= \frac{3\pi}{10}L(f_9, 7) = \frac{3\pi^2}{35}L(f_9, 6) = \frac{4\pi^3}{175}L(f_9, 5) = \frac{\pi^4}{175}L(f_9, 4) \\ &= \frac{\pi^5}{700}L(f_9, 3) = \frac{\pi^6}{2400}L(f_9, 2) = \frac{\pi^7}{5040}L(f_9, 1). \end{aligned}$$

We thank Yifan Yang for pointing out that one can prove the relation

$$L(f_5, 4) = \frac{\pi^2}{5}L(f_5, 2)$$

using Theorem 2.3 in [19].

## 4 Outlook

There are numerous directions for future study. First, motivated by Beukers' and Zagier's numerical investigation of (5), Almkvist, Zudilin [3] and Cooper [16] searched for parameters  $(a, b, c, d)$  such that the three-term relation

$$(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - n(cn^2 + d)u_{n-1}, \quad (43)$$

with initial conditions  $u_{-1} = 0$ ,  $u_0 = 1$ , produces only integer solutions. For  $(a, b, c, d) = (17, 5, 1, 0)$ , we obtain the Apéry numbers  $A(n)$ . In total, there are nine sporadic cases for (43). It is not currently known if each of these cases has an interpolated version which is related (similar to (2)) to the critical  $L$ -value of a modular form of weight 4. Second, we echo the lament in [28] concerning

the lack of algorithmic approaches in directly proving congruences, such as (15), between binomial sums. Third, can one extend the results in [19] to verify the cases in Example 3.7 and, more generally, find an explicit formula for the ratio  $L(f_k, k-1)/L(f_k, 2)$  in terms of a rational number and a power of  $\pi$ ? Fourth, in the context of Section 3, a supercongruence (akin to (36)) has been proven in [27] between the leading coefficient

$$A_{\sigma_8}(n) = \sum_{\substack{k_1, k_2, k_3, k_4=0 \\ k_1+k_2=k_3+k_4}}^n \prod_{i=1}^4 \binom{n}{k_i} \binom{n+k_i}{k_i}$$

and  $\eta(2\tau)^{12}$ , the unique newform in  $S_6(\Gamma_0(4))$ . Does there exist a version of Theorem 3.2 in this case? Fifth, Zudilin [40] recently considered periods of certain instances of rigid Calabi–Yau manifolds, which are expressed in terms of hypergeometric functions. In these instances, he conjecturally indicated a relation between special bases of the hypergeometric differential equations and all critical  $L$ -values of the corresponding modular forms (these relations include those that we observed during the proof of Theorem 2.2). From our present perspective of interpolations of sequences, can one similarly engage all of the critical  $L$ -values? Finally, it would be highly desirable to have a more conceptual understanding of the connection between these (and potentially other) interpolations and  $L$ -values.

## Acknowledgements

The first author would like to thank the Hausdorff Research Institute for Mathematics in Bonn, Germany for their support as this work began during his stay from January 2–19, 2018 as part of the Trimester Program “Periods in Number Theory, Algebraic Geometry and Physics”. He also thanks Masha Vlasenko for her support and encouragement during the initial stages of this project. The authors are particularly grateful to Wadim Zudilin for sharing his proof of Theorem 2.2 for sequence  $\mathbf{F}$ .

## References

- [1] S. Ahlgren. Gaussian hypergeometric series and combinatorial congruences. In *Symbolic computation, number theory, special functions, physics and combinatorics (Gainesville, FL, 1999)*, volume 4 of *Dev. Math.*, pages 1–12. Kluwer Acad. Publ., Dordrecht, 2001.
- [2] S. Ahlgren and K. Ono. A Gaussian hypergeometric series evaluation and Apéry number congruences. *J. Reine Angew. Math.*, 2000(518):187–212, 2000.

- [3] G. Almkvist and W. Zudilin. Differential equations, mirror maps and zeta values. In *Mirror symmetry. V*, volume 38 of *AMS/IP Stud. Adv. Math.*, pages 481–515. Amer. Math. Soc., Providence, RI, 2006.
- [4] G. E. Andrews, R. Askey, and R. Roy. *Special Functions*. Cambridge University Press, Cambridge, 1999.
- [5] R. Apéry. Irrationalité de  $\zeta(2)$  et  $\zeta(3)$ . *Astérisque*, 61:11–13, 1979.
- [6] T. M. Apostol. *Modular functions and Dirichlet series in number theory*. Springer-Verlag, New York, 2nd edition, 1990.
- [7] M.-J. Bertin, A. Feaver, J. Fuselier, M. Lalín, and M. Manes. Mahler measure of some singular  $K3$ -surfaces. In *Women in numbers 2: research directions in number theory*, volume 606 of *Contemp. Math.*, pages 149–169. Amer. Math. Soc., Providence, RI, 2013.
- [8] F. Beukers. A note on the irrationality of  $\zeta(2)$  and  $\zeta(3)$ . *Bull. London. Math. Soc.*, 3(11):268–272, 1979.
- [9] F. Beukers. Another congruence for the Apéry numbers. *J. Number Theory*, 25(2):201–210, 1987.
- [10] F. Beukers. On Dwork’s accessory parameter problem. *Math. Z.*, 241(2):425–444, 2002.
- [11] J. M. Borwein and P. B. Borwein. A cubic counterpart of Jacobi’s identity and the AGM. *Trans. Amer. Math. Soc.*, 323(2):691–701, 1991.
- [12] J. M. Borwein and P. B. Borwein. *Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity*. Wiley, 1998.
- [13] F. Brown. Irrationality proofs for zeta values, moduli spaces and dinner parties. *Mosc. J. Comb. Number Theory*, 6(2–3):102–165, 2016.
- [14] H. H. Chan, Y. Tanigawa, Y. Yang, and W. Zudilin. New analogues of Clausen’s identities arising from the theory of modular forms. *Adv. Math.*, 228(2):1294–1314, 2011.
- [15] H. Cohen. *Number Theory, Volume II: Analytic and Modern Tools*. Springer, New York, 2007.
- [16] S. Cooper. Sporadic sequences, modular forms and new series for  $1/\pi$ . *Ramanujan J.*, 29(1–3):163–183, 2012.
- [17] R. M. Damerell. L-functions of elliptic curves with complex multiplication, I. *Acta Arith.*, 17(3):287–301, 1970.
- [18] S. Formichella and A. Straub. Gaussian binomial coefficients with negative arguments. *Preprint*, Feb. 2018. arXiv:1802.02684.

- [19] S. Fukuhara and Y. Yang. Bases for  $S_k(\Gamma_1(4))$  and formulas for even powers of the Jacobi theta function. *Int. J. Number Theory*, 9(8):1973–1993, 2013.
- [20] A. Hurwitz. Über die Entwicklungskoeffizienten der lemniskatischen Funktionen. *Math. Ann.*, 51:199–226, 1899.
- [21] N. Koblitz. *p-adic Numbers, p-adic Analysis, and Zeta Functions*. Springer, New York, 2nd edition, 1984.
- [22] M. Kontsevich and D. Zagier. Periods. In *Mathematics unlimited—2001 and beyond*, pages 771–808. Springer, Berlin, 2001.
- [23] C. Koutschan. *Advanced Applications of the Holonomic Systems Approach*. PhD thesis, RISC, Johannes Kepler University, Linz, Austria, Sept. 2009.
- [24] J.-J. Lee, M. R. Murty, and D. Park. Generalization of a theorem of Hurwitz. *J. Ramanujan Math. Soc.*, 31(3):215–226, 2016.
- [25] W.-C. W. Li and L. Long. Atkin and Swinnerton-Dyer congruences and noncongruence modular forms. *RIMS Kôkyûroku Bessatsu*, B51:269–299, 2014.
- [26] D. Loeb. Sets with a negative number of elements. *Adv. Math.*, 91(1):64–74, 1992.
- [27] D. McCarthy, R. Osburn, and A. Straub. Sequences, modular forms and cellular integrals. *Preprint*, May 2017. arXiv:1705.05586.
- [28] R. Osburn, A. Straub, and W. Zudilin. A modular supercongruence for  ${}_6F_5$ : An Apéry-like story. *Ann. Inst. Fourier (Grenoble)*, 2018. arXiv:1701.04098.
- [29] A. v. d. Poorten. A proof that Euler missed ... Apéry’s proof of the irrationality of  $\zeta(3)$ . *Math. Intelligencer*, 1(4):195–203, 1979.
- [30] M. Rogers, J. G. Wan, and I. J. Zucker. Moments of elliptic integrals and critical  $L$ -values. *Ramanujan J.*, 37(1):113–130, 2015.
- [31] M. Schütt. CM newforms with rational coefficients. *Ramanujan J.*, 19(2):187–205, 2009.
- [32] A. Selberg and S. Chowla. On Epstein’s zeta-function. *J. Reine Angew. Math.*, 227:86–110, 1967.
- [33] G. Shimura. The special values of the zeta functions associated with cusp forms. *Comm. Pure Appl. Math.*, 29(6):783–804, 1976.
- [34] G. Shimura. On the periods of modular forms. *Math. Ann.*, 229(3):211–221, 1977.

- [35] J. Stienstra and F. Beukers. On the Picard-Fuchs equation and the formal Brauer group of certain elliptic K3-surfaces. *Math. Ann.*, 271(2):269–304, 1985.
- [36] H. A. Verrill. Congruences related to modular forms. *Int. J. Number Theory*, 6(6):1367–1390, 2010.
- [37] D. Zagier. Elliptic modular forms and their applications. In *The 1-2-3 of Modular Forms*. Springer-Verlag, Berlin, 2008.
- [38] D. Zagier. Integral solutions of Apéry-like recurrence equations. In *Groups and symmetries*, volume 47 of *CRM Proc. Lecture Notes*, pages 349–366. Amer. Math. Soc., Providence, RI, 2009.
- [39] D. Zagier. Arithmetic and topology of differential equations. *Proceedings of the 2016 ECM*, 2017.
- [40] W. Zudilin. A hypergeometric version of the modularity of rigid Calabi–Yau manifolds. *Preprint*, May 2018. arXiv:1805.00544.