

A solution of Sun's \$520 challenge concerning $\frac{520}{\pi}$

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Abstract

We prove a Ramanujan-type formula for $520/\pi$ conjectured by Sun. Our proof begins with a hypergeometric representation of the relevant double series, which relies on a recent generating function for Legendre polynomials by Wan and Zudilin. After showing that appropriate modular parameters can be introduced, we then apply standard techniques, going back to Ramanujan, for establishing series for $1/\pi$.

1 Introduction

In [Sun12b], Zhi-Wei Sun offered a \$520 prize for the first correct proof of the following Ramanujan-type formula:

$$\frac{520}{\pi} = \sum_{n=0}^{\infty} \frac{1054n + 233}{480^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{n} (-1)^k 8^{2k-n}. \quad (1)$$

The prize money attached to formula (1) is in honor of the May 20th celebrations at Nanjing University. In this paper, we offer a complete proof of (1).

Our proof may be divided into three parts. First and crucially, in Section 2 we connect series of the form (1) to a product of hypergeometric functions at algebraically related arguments. This first step relies on the recent work [WZ12] of Wan and Zudilin. Secondly, we show in Section 4 that the two algebraic arguments are values of modular functions at simply related quadratic irrationalities. With such a setup, there exist rather standard procedures to

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prove the corresponding series for $1/\pi$ (though we point out that series of the form (23) have only been considered since [CWZ12]). To varying degrees of detail, the approach employed here is, for instance, used in [BB09], [CWZ12] or [WZ12]. For the completeness of this note and for the convenience of the reader, we chose to nevertheless include all necessary details in Section 5. For a historical account and many more references on series for $1/\pi$, originating with [Ram14], we refer to [BB09].

Sun's list of identities [Sun12b] contains many formulas along the lines of (1), which were discovered by computational searches guided by the observation that most series for $1/\pi$ like (1) appear to occur in tandem with congruences for their finite sum analogs. This is illustrated in [Sun12a] where (1) appears as part of Conjecture 7.11. The final Section 6 briefly discusses the applicability of our approach to proving all the identities in [Sun12b, Sun12a] of the same form as (1).

2 Step I: A hypergeometric representation

Since virtually every interesting formula for $1/\pi$ comes from studying hypergeometric functions, we expect that the same should be true for (1). Let us define

$$A(x, y) = \sum_{n=0}^{\infty} x^n \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{n} (-1)^k y^{2k-n}. \quad (2)$$

Clearly then Sun's conjecture (1) is equivalent to

$$233A\left(\frac{1}{480}, 8\right) + 1054(\theta_x A)\left(\frac{1}{480}, 8\right) = \frac{520}{\pi} \quad (3)$$

where $\theta_x = x \frac{d}{dx}$ is the Euler differential operator. Our first step is to reexpress $A(x, y)$ as a series of Legendre polynomials. Recent work of Wan and Zudilin [WZ12] then makes it possible to write $A(x, y)$ as a product of hypergeometric functions.

Proposition 2.1. *Let x and y be such that (2) converges absolutely. Then*

$$A(x, y) = \sum_{k=0}^{\infty} (-xy)^k \binom{2k}{k}^2 P_{2k}\left(\sqrt{1 + \frac{4x}{y}}\right). \quad (4)$$

Proof. If (2) converges absolutely, then we can interchange the order of summa-

tion to obtain

$$\begin{aligned}
A(x, y) &= \sum_{n=0}^{\infty} x^n \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{n} (-1)^k y^{2k-n} \\
&= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} x^n \frac{(2n)!}{k!^2 (n-k)!^2} \frac{(2k)!}{(2k-n)!n!} (-1)^k y^{2k-n} \\
&= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} x^{n+k} \frac{(2n+2k)!}{k!^2 n!^2} \frac{(2k)!}{(k-n)!(n+k)!} (-1)^k y^{k-n} \\
&= \sum_{k=0}^{\infty} (-xy)^k \binom{2k}{k}^2 {}_2F_1 \left(\begin{matrix} 1/2 + k, -k \\ 1 \end{matrix} \middle| -\frac{4x}{y} \right).
\end{aligned}$$

By the quadratic transformation [AAR99, (3.1.3)] for the hypergeometric function, we have

$$\begin{aligned}
{}_2F_1 \left(\begin{matrix} 1/2 + k, -k \\ 1 \end{matrix} \middle| -\frac{4x}{y} \right) &= {}_2F_1 \left(\begin{matrix} 1 + 2k, -2k \\ 1 \end{matrix} \middle| \frac{1 - \sqrt{1 + \frac{4x}{y}}}{2} \right) \\
&= P_{2k} \left(\sqrt{1 + \frac{4x}{y}} \right)
\end{aligned}$$

where $P_k(z)$ is the k th Legendre polynomial. This is (4). \square

The proof of Proposition 2.1 actually provides a convenient criterion for absolute convergence of the original double series (2).

Proposition 2.2. *The double series (2) converges absolutely for x, y if*

$$\left| 16xy \left(\sqrt{\left| \frac{4x}{y} \right|} + \sqrt{1 + \left| \frac{4x}{y} \right|} \right)^2 \right| < 1. \quad (5)$$

Proof. Absolute convergence of $A(x, y)$, as in (2), is equivalent to convergence of

$$A(i|x|, i|y|) = \sum_{n=0}^{\infty} |x|^n \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{n} |y|^{2k-n}.$$

The fact that all summands are positive again justifies interchanging the order of summation as in the proof of Proposition 2.1 to obtain

$$A(i|x|, i|y|) = \sum_{k=0}^{\infty} |xy|^k \binom{2k}{k}^2 P_{2k} \left(\sqrt{1 + \frac{4|x|}{|y|}} \right).$$

The classical asymptotic formula for the Legendre polynomials, e.g. [Sze39, (8.21.1)], states that

$$P_n(z) \cong \frac{\left(z + (z^2 - 1)^{1/2} \right)^{n+1/2}}{(2\pi n)^{1/2} (z^2 - 1)^{1/4}}, \quad (6)$$

as $n \rightarrow \infty$ and is valid for z outside the interval $[-1, 1]$. Here, in the cases of interest, $z > 1$ and combining (6) with the well-known asymptotics for the central binomial coefficients shows (5). \square

In particular, the series (2) converges absolutely for $(x, y) = (\frac{1}{480}, 8)$. Moreover, we find that in this case the convergence of (4) is geometric with ratio $-\frac{64}{225} = -(\frac{8}{15})^2$.

Theorem 2.3. *Let x and y be such that (4) holds. Suppose, further, there are X and Y in a certain neighborhood of 1 such that*

$$-xy = \left(\frac{X - Y}{4(1 + XY)} \right)^2, \quad 1 + \frac{4x}{y} = \left[\frac{(X + Y)(1 - XY)}{(X - Y)(1 + XY)} \right]^2. \quad (7)$$

Then

$$A(x, y) = \frac{1 + XY}{2} {}_2F_1 \left(\begin{matrix} 1/2, 1/2 \\ 1 \end{matrix} \middle| 1 - X^2 \right) {}_2F_1 \left(\begin{matrix} 1/2, 1/2 \\ 1 \end{matrix} \middle| 1 - Y^2 \right). \quad (8)$$

Proof. A series of the form (4) was recently summed by Wan and Zudilin [WZ12]. They proved that when X and Y lie in a certain neighborhood of 1, then

$$\begin{aligned} & \sum_{k=0}^{\infty} \left(\frac{X - Y}{4(1 + XY)} \right)^{2k} \binom{2k}{k}^2 P_{2k} \left(\frac{(X + Y)(1 - XY)}{(X - Y)(1 + XY)} \right) \\ &= \frac{1 + XY}{2} {}_2F_1 \left(\begin{matrix} 1/2, 1/2 \\ 1 \end{matrix} \middle| 1 - X^2 \right) {}_2F_1 \left(\begin{matrix} 1/2, 1/2 \\ 1 \end{matrix} \middle| 1 - Y^2 \right). \end{aligned}$$

If X and Y are chosen as in (7), this is equal to $A(x, y)$ as claimed. \square

Remark 2.4. The solutions to the equations (7) have a remarkable number of symmetries. Namely, if (X, Y) is a solution, then so are $(-X, -Y)$, (X^{-1}, Y^{-1}) , and (Y, X) . Assuming that x and y are real, we also have (\bar{X}, \bar{Y}) as another solution. Less obviously, if (X, Y) is a solution, then so is

$$\left(\frac{1 - X}{1 + X}, \frac{1 - Y}{1 + Y} \right). \quad (9)$$

3 Notation

Before proceeding with the proof of Sun's identity, we introduce some very standard objects from elliptic modular function theory. Unfortunately, notation is not as standard.

Throughout, $q = e^{2\pi i \tau}$ and τ is assumed to lie in the upper half-plane. We denote $D = \frac{1}{2\pi i} \frac{d}{d\tau} = q \frac{d}{dq}$. The Dedekind eta function is

$$\eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n)$$

and its logarithmic derivative determines the Eisenstein series

$$E_2(\tau) = 24 \frac{D\eta}{\eta} = 1 + 24 \sum_{n \geq 1} \frac{nq^n}{1 - q^n}.$$

The standard Jacobi theta functions are

$$\theta_2(\tau) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2/2}, \quad \theta_3(\tau) = \sum_{n=-\infty}^{\infty} q^{n^2/2}, \quad \theta_4(\tau) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2/2}.$$

They can be used to express the elliptic modulus $k(\tau)$ as well as the complementary modulus $k'(\tau)$ as

$$k(\tau) = \left(\frac{\theta_2(\tau)}{\theta_3(\tau)} \right)^2, \quad k'(\tau) = \left(\frac{\theta_4(\tau)}{\theta_3(\tau)} \right)^2.$$

We note the classical $k^2 + (k')^2 = 1$ as well as the transformations

$$k(\tau + 1) = i \frac{k(\tau)}{k'(\tau)}, \quad k'(\tau + 1) = \frac{1}{k'(\tau)}, \quad k\left(-\frac{1}{\tau}\right) = k'(\tau). \quad (10)$$

Finally, the complete elliptic integral of the first kind is

$$K(k) = \frac{\pi}{2} {}_2F_1\left(\begin{matrix} 1/2, 1/2 \\ 1 \end{matrix} \middle| k^2\right)$$

With this setup we have the classical identity

$$\frac{2}{\pi} K(k(\tau)) = {}_2F_1\left(\begin{matrix} 1/2, 1/2 \\ 1 \end{matrix} \middle| k^2(\tau)\right) = \theta_3(\tau)^2, \quad (11)$$

valid in a neighborhood of $i\infty$, which in light of Theorem 2.3 underlies all that follows.

4 Step II: Introducing modular parameters

Proposition 4.1. *If $x = \frac{1}{480}$ and $y = 8$, then*

$$X = k'(\tau_0), \quad Y = k'(5\tau_0),$$

with $\tau_0 = \frac{1}{2} + \frac{3}{10}\sqrt{-5}$, satisfy the two relations (7).

Proof. Note that X and Y are both values of modular functions (with respect to some congruence subgroup) at a quadratic irrational point in the upper half-plane. By standard principles, X and Y are therefore effectively computable algebraic numbers [Cox97], [BB98]. In the case at hand, we find that X and Y both have the palindromic minimal polynomial $z^8 p(z^2 + 1/z^2)$ where

$$p(z) = z^4 + 88796296z^3 + 237562136z^2 - 595063264z - 470492144.$$

Together with the approximate values $X \approx 0.57884718 - 0.81543604i$ and $Y \approx 0.99999998 - 0.00021224i$ this determines both as algebraic numbers. It is then computational routine to verify that the two relations (7) are indeed satisfied. \square

Nevertheless, in order to obtain explicit radical expressions for X and Y , we sketch in Remark 4.2 how one may alternatively use modular equations and known singular values to evaluate $k'(\tau_0)$ and $k'(5\tau_0)$ without assistance by a computer algebra system. Ultimately, this allows us to give rather appealing radical expressions in Remark 4.3.

Remark 4.2. We denote the singular values in the usual way as $k_n = k(i\sqrt{n})$ and $k'_n = k'(i\sqrt{n}) = k(i/\sqrt{n})$. Then Ramanujan's class invariant G_n satisfies (or may be introduced as)

$$G_n^{12} = \frac{1}{2k_n k'_n}.$$

The reason for introducing this additional notation is that the values G_n and k_n (via $\alpha_n := k_n^2$) are extensively tabulated [Ber98, Chapter 34]. We observe that

$$k'(\tau) = \frac{1 - k(2\tau)}{1 + k(2\tau)} \quad (12)$$

is a particular form of the modular equation of degree 2. Hence to find $k'(5\tau_0)$ we need only evaluate $k(10\tau_0)$. To this end, using (10),

$$k(10\tau_0) = k\left(1 + 3i\sqrt{5}\right) = i \frac{k_{45}}{k'_{45}} = 2ik_{45}^2 G_{45}^{12}.$$

Equipped with the values $k_{45}^2 = \alpha_{45}$ and G_{45} found in [Ber98, Chapter 34], we then obtain

$$k(10\tau_0) = i \left(\sqrt{\frac{7 + 3\sqrt{5}}{4}} - \sqrt{\frac{3 + 3\sqrt{5}}{4}} \right)^4 \left(\sqrt{\frac{3 + \sqrt{5}}{2}} - \sqrt{\frac{1 + \sqrt{5}}{2}} \right)^4 \quad (13)$$

and hence have evaluated

$$Y = k'(5\tau_0) = \frac{1 - k(10\tau_0)}{1 + k(10\tau_0)}. \quad (14)$$

Similar considerations apply for $X = k'(\tau_0)$ (which in principle can also be determined from (14) and the modular equation (24) of degree 5).

Remark 4.3. It can be seen from (12) and (9) that the numbers $k'(5\tau_0)$ and $k(10\tau_0)$ are conjugate as algebraic numbers. In light of (13) the evaluation (14)

may therefore be further simplified. Indeed, we have the radical expressions

$$X = i \left(\sqrt{\frac{7-3\sqrt{5}}{4}} - \sqrt{\frac{3-3\sqrt{5}}{4}} \right)^4 \left(\sqrt{\frac{3-\sqrt{5}}{2}} - \sqrt{\frac{1-\sqrt{5}}{2}} \right)^4,$$

$$Y = i \left(\sqrt{\frac{7-3\sqrt{5}}{4}} - \sqrt{\frac{3-3\sqrt{5}}{4}} \right)^4 \left(\sqrt{\frac{3-\sqrt{5}}{2}} + \sqrt{\frac{1-\sqrt{5}}{2}} \right)^4.$$

Each factor in these expressions is an algebraic unit (and hence so are X and Y). In particular, the product XY and the quotient X/Y take the particularly simple forms

$$XY = - \left(\sqrt{\frac{7-3\sqrt{5}}{4}} - \sqrt{\frac{3-3\sqrt{5}}{4}} \right)^8, \quad \frac{X}{Y} = \left(\sqrt{\frac{3-\sqrt{5}}{2}} - \sqrt{\frac{1-\sqrt{5}}{2}} \right)^8.$$

We also note that using the stated radical expressions it is trivial to verify that X and Y satisfy the two relations (7).

Remark 4.4. Let us indicate how one may find the value of τ_0 used in Proposition 4.1. Let $x = \frac{1}{480}$ and $y = 8$ and assume that the numbers X, Y satisfy the two relations (7) which, upon clearing denominators, are polynomial. By computing resultants, we conclude that X and Y both have minimal polynomial $q(z)$ as stated in the proof of Proposition 4.1. For each root X of $q(z)$, we can now find τ_0 such that $X = k'(\tau_0)$ using the classical inversion relation

$$\tau = i \frac{K(k')}{K(k)}$$

with the values $k' = X$ and $k = \sqrt{1 - X^2}$. With τ_0 such determined, $k'(\tau_0)^2 = X^2$. This allows τ_0 to be numerically computed from X and it only remains to recognize τ_0 , as is possible here, as a quadratic irrationality.

Different choices of roots X of $q(z)$ lead to different values for τ_0 but these are related by the symmetries observed in Remark 2.4. Essentially, this leaves two choices: τ_0 as presented in Proposition 4.1 and, related by (12), $\tau_1 = -\frac{1}{10\tau_0}$ which also satisfies the statement of Proposition 4.1. The second choice, however, results in values for X, Y which are not sufficiently close to 1 in order to use (8). This is further explained in Remark 4.6.

Following the notation of [CWZ12] let us denote

$$F(\alpha) = {}_2F_1 \left(\begin{matrix} 1/2, 1/2 \\ 1 \end{matrix} \middle| \alpha \right), \quad G(\alpha) = \alpha \frac{d}{d\alpha} F(\alpha) = \frac{\alpha}{4} {}_2F_1 \left(\begin{matrix} 3/2, 3/2 \\ 2 \end{matrix} \middle| \alpha \right).$$

Throughout, we assume the standard branch cut for F and G , with a cut along the real axis from 1 to ∞ .

Proposition 4.5. *Let $x = \frac{1}{480}$ and $y = 8$. Let $X = k'(\tau_0)$, $Y = k'(5\tau_0)$ for $\tau_0 = \frac{1}{2} + \frac{3}{10}\sqrt{-5}$ as in Proposition 4.1. Then (8) holds in a neighborhood of these values. In particular,*

$$A(x, y) = \frac{1 + XY}{2} F(1 - X^2) F(1 - Y^2). \quad (15)$$

Proof. We note that we cannot rely directly on Theorem 2.3 because X does not lie in a sufficient neighborhood of 1. In fact, $|1 - X^2| \approx 1.63087$ from the approximations for X and Y that were noted in the proof of Proposition 4.1.

Throughout, let $t \in [0, 1)$. We consider the convex combinations

$$X_t = (1 - t)X + t, \quad Y_t = (1 - t)Y + t,$$

along with the corresponding values x_t, y_t , determined by (7). Our strategy is as follows: for $t_0 < 1$ large enough the values X_{t_0}, Y_{t_0} are sufficiently close to 1 so that Theorem 2.3 applies, showing that, for $t = t_0$,

$$A(x_t, y_t) = \frac{1 + X_t Y_t}{2} F(1 - X_t^2) F(1 - Y_t^2). \quad (16)$$

We wish to conclude that (15) holds by analytically continuing (16) from $t = t_0$ to $t = 0$. By construction, X_t, Y_t are not purely imaginary and hence the arguments $1 - X_t^2$ and $1 - Y_t^2$ of F stay off the branch cut. We are thus done if we can show that, for all $t \in [0, t_0]$, the values x_t, y_t are such that the series (4) for $A(x_t, y_t)$ converges absolutely. That this is crucial is illustrated by Remark 4.6.

To begin, we note that

$$1 - X_t Y_t = (1 - t)(1 - XY + t(1 - X)(1 - Y)), \quad (17)$$

$$(1 + X_t Y_t) - (X_t + Y_t) = (1 - t)^2 (1 - X)(1 - Y). \quad (18)$$

Using (17) we then estimate

$$|1 - X_t Y_t| \leq |1 - XY| + |1 - X||1 - Y| \leq 0.92$$

where the final inequality simply holds because with our values of X and Y (recorded in the proof of Proposition 4.1) the middle quantity is, to three digits, equal to 0.918. Consequently,

$$|1 + X_t Y_t| \geq 2 - |1 - X_t Y_t| \geq 1.08,$$

from which we conclude

$$|x_t y_t|^{1/2} = \left| \frac{X_t - Y_t}{4(1 + X_t Y_t)} \right| \leq \frac{|X_t - Y_t|}{4 \cdot 1.08} \leq \frac{|X - Y|}{4 \cdot 1.08} \leq 0.22. \quad (19)$$

Once more, the final inequality depends on our specific values of X and Y . Using both (17) and (18),

$$\begin{aligned}
1 + \frac{4x_t}{y_t} &= \left[\frac{1 - X_t Y_t}{X_t - Y_t} \right]^2 \left[\frac{X_t + Y_t}{1 + X_t Y_t} \right]^2 \\
&= \left[\frac{1 - XY}{X - Y} + \frac{t(1 - X)(1 - Y)}{X - Y} \right]^2 \left[1 - \frac{(1 - t)^2 (1 - X)(1 - Y)}{1 + X_t Y_t} \right]^2 \\
&=: [A + \varepsilon_1]^2 [1 + \varepsilon_2]^2. \tag{20}
\end{aligned}$$

The smallness of the second summands ε_1 and ε_2 is witnessed by

$$|\varepsilon_1| = \left| \frac{t(1 - X)(1 - Y)}{X - Y} \right| \leq \left| \frac{(1 - X)(1 - Y)}{X - Y} \right| \leq 0.00022$$

and

$$|\varepsilon_2| = \left| \frac{(1 - t)^2 (1 - X)(1 - Y)}{1 + X_t Y_t} \right| \leq \frac{|(1 - X)(1 - Y)|}{1.08} \leq 0.00019.$$

Combining these with $|A| \leq 1.00042$, expanding (20), subtracting 1, and using the triangle inequality, we find

$$\left| \frac{4x_t}{y_t} \right| \leq \left| \left(\frac{1 - XY}{X - Y} \right)^2 - 1 \right| + 0.00083 \leq 0.0017. \tag{21}$$

The estimates (19) and (21) together with (5) now show that the series (4) converges absolutely for all pairs x_t, y_t (the estimates show that the convergence is geometric with ratio at most 0.85). \square

Remark 4.6. It follows from (9) and (12) that $\tau_1 = -\frac{1}{10\tau_0}$ also satisfies the statement of Proposition 4.1. However, Proposition 4.5 fails to hold with

$$\begin{aligned}
X &= k'(\tau_1) = k(10\tau_0) \approx 0.000106121305i, \\
Y &= k'(5\tau_1) = k(2\tau_0) \approx 0.51647560i.
\end{aligned}$$

The fact that the corresponding $1 - X^2$ and $1 - Y^2$ are on the branch cut of F is not the reason for the failing of (15), as can be seen from the fact that, for $t > 0$, the $1 - X_t^2, 1 - Y_t^2$ avoid the branch cut so that the monodromy of F is not invoked by ever crossing the cut. However, the values x_t, y_t in the process are such that the domain of convergence of A , as defined in (4), is first left and then reentered again. Thus the failure of (15) is due to the monodromy of A . This illustrates the need for an argument such as the one given in the proof of Proposition 4.5. Clearly, it would be nice to replace it with a more conceptual and concise argument.

5 Step III: The execution

With Proposition 4.5 established, Sun's conjecture (3) may now be confirmed using precisely the techniques of, for instance, [BB09], [CWZ12] and [WZ12]. It is characteristic for this technique to produce unwieldy algebraic numbers along the way. Only in a few cases have we therefore chosen to include these numbers explicitly by giving their minimal polynomial along with a sufficient numerical approximation. Of course, most computer algebra systems are very comfortable with symbolically manipulating algebraic numbers (here, the algebraic degree never exceeds 16) and it is in some way rather fitting that computer algebra is at work when establishing conjectures that were found by computational search.

Proposition 5.1. *Suppose that (8) holds. Then we have*

$$\frac{dA}{dx} = r(X, Y)G(1 - X^2)F(1 - Y^2) + s(X, Y)F(1 - X^2)G(1 - Y^2)$$

where r and s are explicit rational functions.

Proof. We note that

$$\frac{d}{dX}F(1 - X^2) = \frac{-2X}{1 - X^2}G(1 - X^2)$$

and hence, differentiating (8),

$$\begin{aligned} 2\frac{dA}{dx} &= \frac{dA}{dX}\frac{dX}{dx} + \frac{dA}{dY}\frac{dY}{dx} \\ &= \left[YF(1 - X^2) - \frac{2X(1 + XY)}{1 - X^2}G(1 - X^2) \right] F(1 - Y^2)\frac{dX}{dx} \\ &\quad + \left[XF(1 - Y^2) - \frac{2Y(1 + XY)}{1 - Y^2}G(1 - Y^2) \right] F(1 - X^2)\frac{dY}{dx}. \end{aligned}$$

Further, $\frac{dX}{dx}$ and $\frac{dY}{dx}$ are determined by differentiating the two relations (7) which gives

$$\begin{aligned} -y &= \frac{X - Y}{8(1 + XY)^3} \left[(1 + Y^2)\frac{dX}{dx} - (1 + X^2)\frac{dY}{dx} \right], \\ \frac{4}{y} &= \frac{4(X + Y)(1 - XY)}{[(X - Y)(1 + XY)]^3} \left[X(1 + Y^2)(1 - X^2)\frac{dY}{dx} \right. \\ &\quad \left. - Y(1 + X^2)(1 - Y^2)\frac{dX}{dx} \right]. \end{aligned}$$

Combining these results in the claimed combination. \square

For our purposes it will become convenient to employ the algebraic numbers

$$\alpha := 1 - X^2 = k^2(\tau_0), \quad \beta := 1 - Y^2 = k^2(5\tau_0) \quad (22)$$

with τ_0 as in Proposition 4.1. For the convenience of the reader, we record their numerical values as

$$\alpha \approx 1.329871878 + 0.944025712i, \quad \beta \approx 9.00938 \times 10^{-8} + 0.0004244852051i.$$

We can then express the left-hand side of (3) in terms of the functions F and G at the values α and β .

Proposition 5.2. *With α, β as in (22), Conjecture (1) is equivalent to*

$$r_1 F(\alpha)F(\beta) + r_2 G(\alpha)F(\beta) + r_3 F(\alpha)G(\beta) = \frac{520}{\pi} \quad (23)$$

where r_1, r_2, r_3 are explicitly computable algebraic numbers.

Proof. Combining Proposition 4.5 with Proposition 5.1, we find that the left-hand side of (3) equals

$$233 \frac{1 + XY}{2} F(\alpha)F(\beta) + \frac{1054}{480} r(X, Y)G(\alpha)F(\beta) + s(X, Y)F(\alpha)G(\beta)$$

with r, s as in Proposition 5.1. Since X and Y are algebraic, the claim follows. \square

As in [CWZ12, Proposition 3] we next express the values $F(\beta), G(\beta)$ by the corresponding values $F(\alpha), G(\alpha)$.

Proposition 5.3. *Let α, β be as in (22). Then there are explicitly computable algebraic numbers t, t_1, t_2 such that*

- $F(\beta) = tF(\alpha)$,
- $G(\beta) = t_1 F(\alpha) + t_2 G(\alpha)$.

Proof. In the language of Ramanujan, $\beta = k^2(5\tau)$ has degree 5 over $\alpha = k^2(\tau)$, whence the numbers satisfy the modular equation of degree 5 [Ber91, Entry 19.13]

$$(\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} = 1. \quad (24)$$

Differentiating with respect to α yields

$$\begin{aligned} \frac{\beta^{1/2}}{\alpha^{1/2}} - \frac{(1-\beta)^{1/2}}{(1-\alpha)^{1/2}} + 2 \frac{1-2\alpha}{3} \frac{[16\beta(1-\beta)]^{1/6}}{[\alpha(1-\alpha)]^{5/6}} \\ + \left[\frac{\alpha^{1/2}}{\beta^{1/2}} - \frac{(1-\alpha)^{1/2}}{(1-\beta)^{1/2}} + 2 \frac{1-2\beta}{3} \frac{[16\alpha(1-\alpha)]^{1/6}}{[\beta(1-\beta)]^{5/6}} \right] \frac{d\beta}{d\alpha} = 0, \end{aligned}$$

which, upon taking the appropriate roots, provides us with $\frac{d\beta}{d\alpha}$ as an algebraic function of α .

Moreover, the multiplier $\frac{F(\alpha)}{F(\beta)}$ may be expressed as [Ber91, Entry 19.13]

$$\frac{F(\alpha)}{F(\beta)} = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}, \quad (25)$$

which verifies the first part of the claim.

Further, taking the logarithmic derivative of (25) with respect to α , we find

$$\frac{G(\alpha)}{\alpha F(\alpha)} - \frac{G(\beta)}{\beta F(\beta)} \frac{d\beta}{d\alpha} = \frac{d}{d\alpha} \left[\left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4} \right].$$

Noting that $\frac{d\beta}{d\alpha}$ as well as the right-hand side are algebraic functions of α , the second claim follows from the first. \square

Remark 5.4. We briefly note that Proposition 5.3 holds in much greater generality. For instance, let $\alpha = k^2(\tau)$ and $\beta = k^2(M \cdot \tau)$ where $M \in \text{GL}_2(\mathbb{Z})$ is an integral matrix acting by a linear fractional transformation. Then α, β are both modular functions (with respect to some congruence subgroup) and hence algebraically related (as made explicit by (24) when M acts by multiplication with 5). Moreover, because of (11), $F(\alpha)$ and $F(\beta)$ are both modular forms of weight 1. Hence the quotient $F(\alpha)/F(\beta)$ is a modular function and again an algebraic function of α (as illustrated by (25)). Thus if τ is a quadratic irrationality, Proposition 5.3 holds for α, β as well.

At this stage, we have expressed Sun's conjecture (1) in the form

$$r_1 F(\alpha)^2 + r_2 F(\alpha)G(\alpha) = \frac{520}{\pi} \quad (26)$$

where r_1, r_2 are explicit algebraic numbers.

Proposition 5.5. *With $\alpha = k^2(\tau)$, we have*

$$F(\alpha)G(\alpha) = \frac{E_2(\tau) + (2\alpha - 1)F(\alpha)^2}{6(1 - \alpha)}.$$

Proof. This is [Ber91, Entry 17.9(iv)] in mildly different notation. \square

Applying Proposition 5.5 with α as in (22), equation (26) becomes

$$s_1 F(\alpha)^2 + 52\sqrt{5}E_2(\tau_0) = \frac{520}{\pi} \quad (27)$$

where $s_1 \approx 5.0538411 - 7.1194683i$ is an algebraic number of degree 8. To be very explicit, $\frac{s_1}{52\alpha}$ has minimal polynomial

$$x^8 + 14197606x^6 - 56569153x^4 + 15962594x^2 + 175561.$$

In the final step, we relate, in a standard way, the Eisenstein series $E_2(\tau)$ at our specific point of interest to an algebraic combination of $\frac{1}{\pi}$ and $\theta_3(\tau)^4$. The fact that this is possible whenever τ is a quadratic irrationality is what lies behind the abundance of series for $\frac{1}{\pi}$.

Proposition 5.6. *With $\alpha = k^2(\tau_0)$ and $\tau_0 = \frac{1}{2} + \frac{3}{10}\sqrt{-5}$ as in Proposition 4.1, we have*

$$E_2(\tau_0) = \frac{2\sqrt{5}}{\pi} + s_2 F(\alpha)^2 \quad (28)$$

where s_2 is an algebraic number explicited at the end of the proof.

Proof. The main tool of this evaluation is the fact that Ramanujan's multiplier of the second kind, given by [BB98, Chapter 5]

$$R_p(l, k) := \frac{pE_2(p\tau) - E_2(\tau)}{\theta_3^2(p\tau)\theta_3^2(\tau)}, \quad (29)$$

is an algebraic function of $l := k(p\tau)$ and $k := k(\tau)$ (and hence an algebraic number when τ is a quadratic irrationality). For our specific value $\tau = \tau_0$ we will have use of the three instances

$$\begin{aligned} R_2(l, k) &= l' + k, \\ R_3(l, k) &= 1 + kl + k'l', \\ R_5(l, k) &= (3 + kl + k'l') \sqrt{\frac{1 + kl + k'l'}{2}}. \end{aligned}$$

By differentiating the transformation law for the Dedekind eta function, we also have

$$E_2\left(-\frac{1}{\tau}\right) = \tau^2 E_2(\tau) + \frac{6\tau}{\pi i}. \quad (30)$$

If $\tau = i\sqrt{p}$ then $-1/\tau = i/\sqrt{p} = \tau/p$. In that case (29) and (30) give two relations between $E_2(\tau)$ and $E_2(\tau/p)$. Also note that, for such τ , we have $l = k'$ and $l' = k$ by (10). In the case, $p = 5$ we thus obtain

$$\begin{aligned} E_2\left(i/\sqrt{5}\right) &= \frac{3\sqrt{5}}{\pi} - \frac{1}{2\sqrt{5}}\theta_3\left(i/\sqrt{5}\right)^4 (3 + 2kk') \sqrt{\frac{1 + 2kk'}{2}} \\ &= \frac{3\sqrt{5}}{\pi} - \frac{\sqrt{1 + \sqrt{5}}}{\sqrt{10}}\theta_3\left(i/\sqrt{5}\right)^4. \end{aligned} \quad (31)$$

where in the second step we used the singular values [BB98, Chapter 4]

$$\begin{aligned} k\left(i\sqrt{5}\right) &= \frac{\sqrt{\sqrt{5} - 1} - \sqrt{3 - \sqrt{5}}}{2}, \\ k'\left(i\sqrt{5}\right) &= \frac{\sqrt{\sqrt{5} - 1} + \sqrt{3 - \sqrt{5}}}{2}. \end{aligned}$$

Using the modular equations of degree 3 (of both first and second kind), we now find an algebraic r_1 such that

$$3E_2\left(3i/\sqrt{5}\right) - E_2\left(i/\sqrt{5}\right) = r_1\theta_3\left(3i/\sqrt{5}\right)^4$$

and hence, by (31), an algebraic number r_2 such that

$$E_2\left(1 + 3i/\sqrt{5}\right) = E_2\left(3i/\sqrt{5}\right) = \frac{\sqrt{5}}{\pi} + r_2\theta_3\left(3i/\sqrt{5}\right)^4.$$

A further and likewise application of the modular equations of degree 2 yields

$$E_2\left(\frac{1}{2} + \frac{3}{10}\sqrt{-5}\right) = \frac{2\sqrt{5}}{\pi} + s_2\theta_3\left(\frac{1}{2} + \frac{3}{10}\sqrt{-5}\right)^4$$

where $s_2 \approx -0.043464355 + 0.061229289i$ has minimal polynomial

$$625x^8 - 47597450000x^7 + 64879599000x^6 + 34024656000x^5 \\ - 58306698000x^4 + 168524800x^3 + 8089408640x^2 + 722959360x + 44943616.$$

In light of (11), this is the claimed (28). \square

Theorem 5.7. *Sun's conjecture (1) holds true.*

Proof. We combine the left-hand side of (27) with (28) to get

$$s_1F(\alpha)^2 + 52\sqrt{5}E_2(\tau_0) = \left(s_1 + 52\sqrt{5}s_2\right)F(\alpha)^2 + \frac{520}{\pi}.$$

The claim follows by verifying that $s_1 + 52\sqrt{5}s_2 = 0$. \square

6 Further series of the same form

The other conjectures in [Sun12b] of the same form as (1) can be proved analogously to (1). We therefore only provide the counterpart of Proposition 4.1.

Proposition 6.1. *Let x, y, τ, p be as in the table below. Then*

$$X = k'(\tau), \quad Y = k'(p\tau),$$

satisfy the two relations (7). Moreover, equation (8) holds.

In the cases, where p occurs starred, $Y = k'(p\tau + 1) = 1/k'(p\tau)$ has to be taken instead.

Theorem 6.2. *Identities (3.11) to (3.27) in [Sun12b] hold.*

Proof. Using Proposition 6.1 in place of Proposition 4.1, the proofs are analogous to the proof of (1) which was completed in Theorem 5.7. (The starred cases need some slight adjustments to Proposition 5.3 which continues to hold in light of Remark 5.4.) \square

Remark 6.3. Let $T_k(b, c)$ denote, as in [Sun12b], the coefficient of x^k in the expansion of $(x^2 + bx + c)^k$. In terms of the Legendre polynomials, these quantities can be expressed as [Sun12c]

$$T_k(b, c) = (b^2 - 4c)^{k/2} P_k\left(\frac{b}{\sqrt{b^2 - 4c}}\right).$$

x	y	τ	p	[Sun12b]	Remark 6.3
$\frac{i}{240}$	$6i$	$i\sqrt{\frac{3}{10}}$	5	(3.11)	(IV5)
$\frac{i}{289}$	$14i$	$i\sqrt{\frac{1}{7}}$	7	(3.12)	[WZ12, (33)]
$\frac{i}{2800}$	$14i$	$i\sqrt{\frac{7}{10}}$	5	(3.13)	(IV11)
$\frac{i}{576}$	$21i$	$i\sqrt{\frac{3}{14}}$	7	(3.14)	(IV9)
$\frac{i}{46800}$	$36i$	$i\sqrt{\frac{13}{10}}$	5	(3.15)	(IV13)
$\frac{i}{2304}$	$45i$	$i\sqrt{\frac{5}{14}}$	7	(3.16)	(IV10)
$\frac{i}{439280}$	$76i$	$i\sqrt{\frac{19}{10}}$	5	(3.17)	(IV17)
$-\frac{i}{29584}$	$175i$	$\frac{1}{2} + \frac{i}{2}\sqrt{\frac{19}{7}}$	7	(3.18)	(IV21)
$\frac{i}{5616}$	$300i$	$i\sqrt{\frac{3}{26}}$	13	(3.19)	(IV14)
$\frac{i}{28880}$	$1156i$	$i\sqrt{\frac{5}{26}}$	13	(3.20)	(IV16)
$\frac{i}{20400}$	$1176i$	$i\sqrt{\frac{3}{34}}$	17	(3.21)	(IV15)
$\frac{i}{243360}$	$12321i$	$i\sqrt{\frac{5}{38}}$	19	(3.22)	(IV18)
$\frac{1}{48}$	$\frac{15}{16}$	$\frac{1}{2} + \frac{i}{2}\sqrt{\frac{5}{3}}$	3^*	(3.23)	(IV1)
$\frac{1}{480}$	8	$\frac{1}{2} + \frac{3i}{2}\sqrt{\frac{1}{5}}$	5	(3.24)	(IV2)
$\frac{1}{5760}$	18	$\frac{1}{2} + \frac{i}{2}\sqrt{\frac{17}{5}}$	5	(3.25)	(IV3)
$-\frac{1}{48}$	$\frac{9}{8}$	$i\sqrt{\frac{1}{3}}$	3^*	(3.26)	(IV19)
$-\frac{1}{288}$	$\frac{225}{16}$	$i\sqrt{\frac{1}{7}}$	7^*	(3.27)	(IV20)

Table 1: The modular parametrization for the additional entries from [Sun12b]

Using (4) we therefore have

$$A(x, y) = \sum_{k=0}^{\infty} (-x^2)^k \binom{2k}{k}^2 T_{2k} \left(\sqrt{4 + \frac{y}{x}}, 1 \right).$$

For instance,

$$A\left(\frac{1}{480}, 8\right) = \sum_{k=0}^{\infty} \frac{1}{(-480^2)^k} \binom{2k}{k}^2 T_{2k}(62, 1)$$

which we notice as a building block for entry (IV2) in [Sun12b, Sun12c],

$$\sum_{k=0}^{\infty} \frac{340k + 59}{(-480^2)^k} \binom{2k}{k}^2 T_{2k}(62, 1) = \frac{120}{\pi}, \quad (32)$$

thus connecting entries (3.24) and (IV2). Similar relations exist for all of the cases considered in Proposition 6.1 and for each case we have recorded the corresponding entry of [Sun12b] in Table 1 (in the second case the sum was discovered in [WZ12]). We remark that the cases in group (IV) not thus connected by Table 1 correspond to series arising from (2) where $y = i$.

We remark that the equivalence of (32) and (1) relies on

$$2A\left(\frac{1}{480}, 8\right) - 28(\theta_x A)\left(\frac{1}{480}, 8\right) + 65(\theta_y A)\left(\frac{1}{480}, 8\right) = 0 \quad (33)$$

where $\theta_z = z \frac{d}{dz}$ is the Euler differential operator. The truth of (33) follows from the truth of (1) and (32), which was proved in [WZ12]. In fact, as pointed out by James Wan, (33) can be obtained by combining (32) and its “companion series”, see [CWZ12, Section 7] and [WZ12, Remark 2]. Underlying the concept of a companion series is that when one has two parameters, such as x, y , one can differentiate with respect to either of them. For our purposes only the derivative with respect to x was considered but, clearly, Proposition 5.1 works equally well for $\frac{dA}{dy}$. Pursuing this, one obtains another series for $1/\pi$, a companion of the original series.

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