

ON A SECANT DIRICHLET SERIES AND EICHLER INTEGRALS OF EISENSTEIN SERIES

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ABSTRACT. We consider, for even s , the secant Dirichlet series $\psi_s(\tau) = \sum_{n=1}^{\infty} \frac{\sec(\pi n \tau)}{n^s}$, recently introduced and studied by Lalín, Rodrigue and Rogers. In particular, we show, as conjectured and partially proven by Lalín, Rodrigue and Rogers, that the values $\psi_{2m}(\sqrt{r})$, with $r > 0$ rational, are rational multiples of π^{2m} . We then put the properties of the secant Dirichlet series into context by showing that they are Eichler integrals of odd weight Eisenstein series of level 4. This leads us to consider Eichler integrals of general Eisenstein series and to determine their period polynomials. In the level 1 case, these polynomials were recently shown by Murty, Smyth and Wang to have most of their roots on the unit circle. We provide evidence that this phenomenon extends to the higher level case. This observation complements recent results by Conrey, Farmer and Imamoglu as well as El-Guindy and Raji on zeros of period polynomials of Hecke eigenforms in the level 1 case. Finally, we briefly revisit results of a similar type in the works of Ramanujan.

1. INTRODUCTION

Our considerations begin with the secant Dirichlet series

$$(1) \quad \psi_s(\tau) = \sum_{n=1}^{\infty} \frac{\sec(\pi n \tau)}{n^s},$$

which were recently introduced and studied by Lalín, Rodrigue and Rogers [LRR14]. One of the motivations for considering these sums is their similarity in shape and, as we will see, in properties to the cotangent Dirichlet series

$$(2) \quad \xi_s(\tau) = \sum_{n=1}^{\infty} \frac{\cot(\pi n \tau)}{n^s}.$$

For instance, as first proved by Lerch and also recorded by Ramanujan (see Section 8 as well as [Ber77] or [Ber89, p. 276]) the difference $\tau^{2m} \xi_{2m+1}(-1/\tau) - \xi_{2m+1}(\tau)$ is a rational function in τ . This modular functional equation and its ramifications continue to inspire research to this date, including, for instance, [GMR11], [MSW11], [Riv12], [LS13].

As shown in [LRR14], the secant Dirichlet series $\psi_{2m}(\tau)$ satisfy modular functional equations as well. In Section 2, we give an alternative derivation of these functional equations based on the residue theorem and in the spirit of [Ber76]. In this way, we obtain a compact representation of the associated rational function as a certain Taylor coefficient of a quotient of trigonometric functions. We then show

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in Section 3, as conjectured and partially proven in [LRR14], that, for a positive rational number r , $\psi_{2m}(\sqrt{r})$ is a rational multiple of π^{2m} .

In Section 4, we observe that the $2m$ th derivative of $\psi_{2m}(2\tau)$ is, up to a constant term, an Eisenstein series of weight $2m + 1$, level 4 and character χ_{-4} . In other words, the secant Dirichlet series ψ_{2m} are Eichler integrals of Eisenstein series; the basic theory of Eichler integrals is reviewed in Section 5. In this light, several of the properties of the function ψ_{2m} become natural and, in Section 6, we use the modular setting to give another derivation of the functional equation by evaluating the period polynomial of the corresponding Eisenstein series.

In fact, the computation of the period polynomials in Section 6 is carried out more generally for Eisenstein series corresponding to pairs of Dirichlet characters. A special case of this computation allows us, for instance, to rederive the Ramanujan-style formulas for Dirichlet L -values of [Kat74]. The resulting period polynomials mirror the well-known polynomials occurring in the level 1 case, studied, for instance, in [GMR11] and [MSW11], where they (more accurately, their odd parts) are referred to as the *Ramanujan polynomials*. In [MSW11] it was shown that the Ramanujan polynomials are nearly unimodular, that is, all their nonreal roots lie on the unit circle. On the other hand, it was conjectured in [LR13], and proven in [LS13], that the full period polynomial is unimodular in the level 1 case, and this property is also shown to hold for several related polynomials. We indicate in Section 7 that, after a linear change of variables, the period polynomials in higher level appear to have all or most of their roots on the unit circle as well. This observation fits well with and complements the recent result of [CFI12] and [EGR13], where it is shown that the nontrivial zeros of period polynomials of modular forms, which are Hecke eigenforms of level 1, all lie on the unit circle. As an application, we derive formulas for Dirichlet L -values in terms of values of an Eichler integral at algebraic arguments of modulus 1, thus generalizing the formulas for $\zeta(2k + 1)$ studied in [GMR11].

Finally, in Section 8, we return to and discuss related entries in Ramanujan's notebook [Ber89], including the particularly famous entry corresponding to the cotangent Dirichlet series (2), which, for odd s , are the Eichler integrals of the Eisenstein series for the full modular group. In particular, we close by demonstrating that the functional equation for $\psi_{2m}(\tau)$ is in fact a consequence of an identity stated by Ramanujan.

It is on purpose, and hopefully to the benefit of some readers, that the considerations in this paper start out entirely explicit and elementary, with theoretical background, such as Eisenstein series of higher level and Eichler integrals, being included as we proceed. As a consequence of this approach, some of the earlier results can be obtained as special cases of later results.

Convergence of series such as (1) and (2), when τ is a real number, is a rather subtle issue; see, for instance, [Riv12]. It is shown in [LRR14] that $\psi_s(\tau)$, for $s \geq 2$, converges absolutely for rational τ with odd denominator as well as for algebraic irrational τ . On the other hand, the series converges absolutely for all nonreal τ , and our discussion of the modular properties proceeds under the tacit assumption that τ is not real. In the case of evaluations of $\psi_{2m}(\tau)$ at real quadratic τ in Section 3, one may then use limiting arguments to show that the functional equations also hold for these arguments.

2. THE FUNCTIONAL EQUATION OF THE SECANT DIRICHLET SERIES VIA RESIDUES

Obviously, $\psi_s(\tau)$ is periodic of period 2, that is $\psi_s(\tau+2) = \psi_s(\tau)$. In this section, we present an alternative proof of the additional nontrivial functional equation satisfied by (1) in the case $s = 2m$. In [LRR14] this functional equation was established by skillfully and carefully convoluting partial fraction expansions. The proof given here is based on the residue theorem and is in the spirit of the proof given in [Ber76] for the cotangent Dirichlet series (2). Throughout, we denote with $[z^n]f(z)$ the n th coefficient of the Taylor expansion of $f(z)$.

Theorem 2.1. *Let $m > 0$ be an integer. Then we have*

$$(3) \quad \begin{aligned} & (1 + \tau)^{2m-1} \psi_{2m} \left(\frac{\tau}{1 + \tau} \right) - (1 - \tau)^{2m-1} \psi_{2m} \left(\frac{\tau}{1 - \tau} \right) \\ &= \pi^{2m} [z^{2m-1}] \frac{\sin(\tau z)}{\sin((1 - \tau)z) \sin((1 + \tau)z)}. \end{aligned}$$

Proof. Let $s > 0$ be an integer and consider

$$(4) \quad I_N = \frac{1}{2\pi i} \int_{C_N} \frac{\sin\left(\pi \frac{b-a}{2} z\right)}{\sin(\pi a z) \sin(\pi b z)} \frac{dz}{z^{s+1}},$$

where C_N is a positively oriented circle of radius R_N centered at the origin. As in [Ber76], the radii R_N are chosen such that the points on the circle are always at least some fixed positive distance from any of the points $\frac{n}{a}$ and $\frac{n}{b}$, where n ranges over the integers. It is then easily seen that

$$(5) \quad \lim_{N \rightarrow \infty} I_N = 0.$$

The integrand of (4) has poles at $z = 0$ as well as at $z = \frac{n}{a}$ and $z = \frac{n}{b}$ for $n \in \mathbb{Z}$. Writing $\text{Res}(\alpha)$ for the residue of the integrand at $z = \alpha$, we have

$$\text{Res}\left(\frac{n}{a}\right) = a^s \frac{(-1)^n}{\pi} \frac{1}{n^{s+1}} \frac{\sin\left(\pi n \frac{b-a}{2a}\right)}{\sin\left(\pi n \frac{b}{a}\right)} = \frac{a^s \sec\left(\pi n \frac{b-a}{2a}\right)}{2\pi n^{s+1}},$$

where the last equality is obtained by writing $(-1)^n = \frac{1}{\cos(\pi n)}$ and using the trigonometric identity

$$\cos(x) \sin(y) - \cos(y) \sin(x) = 2 \cos\left(\frac{y-x}{2}\right) \sin\left(\frac{y+x}{2}\right),$$

with $x = \pi n$ and $y = \pi n \frac{b}{a}$. By symmetry,

$$\text{Res}\left(\frac{n}{b}\right) = -\frac{b^s \sec\left(\pi n \frac{b-a}{2b}\right)}{2\pi n^{s+1}}.$$

For the residue at the origin, we find that

$$\text{Res}(0) = [z^s] \frac{\sin\left(\pi \frac{b-a}{2} z\right)}{\sin(\pi a z) \sin(\pi b z)} = \pi^s [z^s] \frac{\sin\left(\frac{b-a}{2} z\right)}{\sin(az) \sin(bz)} =: \pi^s p_s(a, b).$$

We now assume that s is odd. Collecting residues and letting $N \rightarrow \infty$, we obtain, using (5),

$$0 = \frac{a^s}{\pi} \sum_{n=1}^{\infty} \frac{\sec\left(\pi n \frac{b-a}{2a}\right)}{n^{s+1}} - \frac{b^s}{\pi} \sum_{n=1}^{\infty} \frac{\sec\left(\pi n \frac{b-a}{2b}\right)}{n^{s+1}} + \pi^s p_s(a, b).$$

Finally, let $a = 1 - \tau$ and $b = 1 + \tau$, and write $s = 2m - 1$, to obtain the functional equation in the form

$$(1 + \tau)^{2m-1} \psi_{2m} \left(\frac{\tau}{1 + \tau} \right) - (1 - \tau)^{2m-1} \psi_{2m} \left(\frac{\tau}{1 - \tau} \right) = \pi^{2m} p_{2m-1}(1 - \tau, 1 + \tau),$$

as claimed. \square

Note that, upon replacing $\tau \rightarrow \frac{\tau}{\tau+1}$ and simplifying slightly, we find that equation (3) becomes

$$(6) \quad (2\tau + 1)^{2m-1} \psi_{2m} \left(\frac{\tau}{2\tau + 1} \right) - \psi_{2m}(\tau) = \pi^{2m} [z^{2m-1}] \frac{\sin(\tau z)}{\sin(z) \sin((2\tau + 1)z)},$$

which, while less symmetric, makes the modular transformation property more apparent.

Remark 2.2. The rational functions on the right-hand sides of (3) and (6) can be made more explicit by expanding the sine functions. Namely, if we expand the two sine functions in the denominator using the defining generating function,

$$(7) \quad \frac{ze^{xz}}{e^z - 1} = \sum_{n \geq 0} B_n(x) \frac{z^n}{n!},$$

of the Bernoulli polynomials $B_n(x)$, then the rational functions are seen to be equal to double sums such as

$$[z^{2m-1}] \frac{\sin\left(\frac{b-a}{2}z\right)}{\sin(az) \sin(bz)} = \frac{(-1)^m}{ab} \sum_{k+n+r=m} \frac{\left(\frac{b-a}{2}\right)^{2k+1} B_{2n}\left(\frac{1}{2}\right) B_{2r}\left(\frac{1}{2}\right) (2a)^{2n} (2b)^{2r}}{(2k+1)!(2n)!(2r)!},$$

where the sum is over nonnegative integers k, n, r summing to m . We remark that $B_n\left(\frac{1}{2}\right) = -(1 - 2^{1-n})B_n$, where $B_n = B_n(0)$ is the n th Bernoulli number. Note that this shows, in particular, that the right-hand sides of equations (3) and (6) are rational functions of the form $\pi^{2m} p_{2m}(\tau)/(1 - \tau^2)$ and $\pi^{2m} q_{2m}(\tau)/(2\tau + 1)$, respectively, where p_{2m} and q_{2m} are polynomials of degree $2m + 1$ with rational coefficients.

In fact, as we will see from versions of the functional equation derived later in this paper, the mentioned double sums can be reduced to single summations; see Remark 6.7.

3. THE SECANT DIRICHLET SERIES AT REAL QUADRATIC IRRATIONALS

In [LRR14] it was conjectured and partially proven that $\psi_{2m}(\sqrt{k})$ is a rational multiple of π^{2m} whenever m and k are positive integers. In this section, we prove this assertion and extend it to the case when k is a rational number. For the sake of simplicity, we first prove, in Theorem 3.2, the case of integral k and then briefly indicate how the more general case, stated in Theorem 3.4, follows in essentially the same way. We point out that an independent proof, based on the theory of generalized η -functions, of Theorem 3.4 has been given by P. Charollois and M. Greenberg [CG14]. We thank M. Lalín, who received preprints of both [CG14] and this paper within a couple of days, for making us aware of this reference.

Consider the matrices

$$(8) \quad A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

As usual, we denote by $\Gamma(2)$ the congruence subgroup

$$\Gamma(2) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv I \pmod{2}\},$$

where I is the identity matrix. The group $\Gamma(2)$ is generated by A , B and $-I$; see, for instance, [Yos97, Exercise II.6, p.34].

Remark 3.1. We note that, as is well-known, $\Gamma(2)$ is conjugate to $\Gamma_0(4)$ under $\tau \mapsto 2\tau$, and that in this setting the group $\langle A, B \rangle$ corresponds to $\Gamma_1(4)$.

In terms of the matrices A and B , we then have

$$(9) \quad \psi_{2m}(A\tau) = \psi_{2m}(\tau), \quad \psi_{2m}(B\tau) = \frac{1}{(2\tau+1)^{2m-1}} \psi_{2m}(\tau) + \pi^{2m} f_{2m}(\tau),$$

where $f_{2m}(\tau)$, given in (6), is a rational function over \mathbb{Q} .

Note that the first equation in (9) simply expresses the periodicity of ψ_{2m} , that is $\psi_{2m}(\tau+2) = \psi_{2m}(\tau)$, while the second one is the functional equation (6) discussed in the previous section.

Theorem 3.2. *Let m and k be positive integers. Then*

$$\psi_{2m}(\sqrt{k}) \in \pi^{2m} \mathbb{Q}.$$

Proof. Following [LRR14], we observe that if the integers X and Y solve Pell's equation

$$(10) \quad X^2 - kY^2 = 1,$$

then $D \cdot \sqrt{k} = \sqrt{k}$, where D is the matrix

$$D = \begin{pmatrix} X & kY \\ Y & X \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Here, and in the sequel, we let 2×2 matrices act on complex numbers by linear fractional transformations and write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{ax+b}{cx+d}.$$

A proof that for every positive nonsquare k there exist nontrivial solutions X, Y to Pell's equation (10) was first published by Lagrange in 1768 [Lag92]. For further information and background on Pell's equation we refer to [Len02].

We now make the simple observation that, by (10),

$$D^2 = \begin{pmatrix} X & kY \\ Y & X \end{pmatrix}^2 = \begin{pmatrix} X^2 + kY^2 & 2kXY \\ 2XY & X^2 + kY^2 \end{pmatrix} \in \Gamma(2).$$

Hence, we can always find $C \in \langle A, B, -I \rangle = \Gamma(2)$, $C \neq \pm I$, such that $C \cdot \sqrt{k} = \sqrt{k}$.

Let $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be such a matrix. Note that necessarily $c \neq 0$.

Repeatedly applying the functional equations (9), we deduce that

$$\psi_{2m}(\sqrt{k}) = \psi_{2m}(C \cdot \sqrt{k}) = \frac{1}{(c\sqrt{k}+d)^{2m-1}} \psi_{2m}(\sqrt{k}) + \pi^{2m} f_{2m,C}(\sqrt{k}),$$

where $f_{2m,C}(\tau) \in \mathbb{Q}(\tau)$ is a rational function with rational coefficients. Consequently,

$$\psi_{2m}(\sqrt{k}) = \pi^{2m} g_k(\sqrt{k})$$

for some rational function $g_k(\tau) \in \mathbb{Q}(\tau)$. On the other hand, the above argument applied to $-\sqrt{k}$ shows that $\psi_{2m}(-\sqrt{k}) = \pi^{2m}g_k(-\sqrt{k})$. Since ψ_{2m} is even, it follows that

$$g_k(\sqrt{k}) = g_k(-\sqrt{k}),$$

which implies that $g_k(\sqrt{k})$ is a rational number. \square

Example 3.3. We now illustrate Theorem 3.2 by evaluating $\psi_2(\sqrt{2})$. In this case, the fundamental solution to Pell's equation (10) is given by $(X, Y) = (3, 2)$, obtained from the first two terms of the continued fraction of $\sqrt{2}$. Then

$$C = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} = -AB^{-1}A,$$

and, indeed, one easily verifies that $\sqrt{2}$ is fixed under $AB^{-1}A$. In the present case $s = 1$, and the transformations (9) satisfied by ψ_2 take the form

$$(11) \quad \psi_2(A\tau) = \psi_2(\tau), \quad \psi_2(B\tau) = \frac{1}{2\tau+1}\psi_2(\tau) + \pi^2 f_2(\tau),$$

with

$$f_2(\tau) = \frac{\tau(3\tau^2 + 4\tau + 2)}{6(2\tau + 1)^2}.$$

We therefore find that

$$\begin{aligned} \psi_2(AB^{-1}A\tau) &= \psi_2(B^{-1}A\tau) \\ &= (2B^{-1}A\tau + 1) [\psi_2(A\tau) - \pi^2 f_2(B^{-1}A\tau)] \\ &= -\frac{1}{2\tau+3}\psi_2(\tau) - \frac{(\tau+2)(3\tau^2+8\tau+6)}{6(2\tau+3)^2}\pi^2. \end{aligned}$$

For the second equality we applied the second transformation of (11) with $B^{-1}A\tau$ in place of τ , while for the third equality we use the fact that $B^{-1}A\tau = -\frac{\tau+2}{2\tau+3}$.

For $\tau = \sqrt{2}$ this reduces to

$$\psi_2(\sqrt{2}) = (2\sqrt{2} - 3)\psi_2(\sqrt{2}) + \frac{2}{3}(\sqrt{2} - 2)\pi^2,$$

which has the solution $\psi_2(\sqrt{2}) = -\frac{\pi^2}{3}$, in agreement with the value given in [LRR14]. Families of more general explicit evaluations of ψ_{2m} at real quadratic irrationalities are derived in Example 6.8.

In fact, it is the case that $\psi_{2m}(\sqrt{r})$ is a rational multiple of π^{2m} whenever r is a rational number. This may be shown in essentially the same fashion, as we indicate next.

Theorem 3.4. *Let m be a positive integer and r a positive rational number such that \sqrt{r} is not a rational number with an even denominator. Then*

$$\psi_{2m}(\sqrt{r}) \in \pi^{2m}\mathbb{Q}.$$

Proof. Write $r = \frac{a}{b}$, where a and b are positive integers. Similarly to the proof of Theorem 3.2, we observe that if the integers X and Y solve Pell's equation $X^2 - abY^2 = 1$, then $D \cdot \sqrt{r} = \sqrt{r}$, where D is the matrix

$$D = \begin{pmatrix} X & aY \\ bY & X \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Again, one observes that $D^2 \in \Gamma(2)$. Hence we can find $C \in \langle A, B, -I \rangle = \Gamma(2)$, $C \neq \pm I$, such that $C \cdot \sqrt{r} = \sqrt{r}$.

It remains to proceed exactly as in the proof of Theorem 3.2. \square

Remark 3.5. We remark that (parts of) the proofs of Theorems 3.2 and 3.4 also apply to other functions that satisfy a modular transformation which is of a similar form as the one for the secant Dirichlet series ψ_{2m} . This includes the class of Eichler integrals to which, as discussed in the Section 5, ψ_{2m} belongs.

4. EISENSTEIN SERIES

We begin this section by observing a natural relation between the secant Dirichlet series ψ_{2m} and Eisenstein series. Specifically, we note that the $2m$ th derivative of ψ_{2m} is, essentially, an Eisenstein series of weight $2m+1$ and level 4. In the language of Section 5, this means that the secant Dirichlet series are Eichler integrals of certain Eisenstein series. In order to investigate Eichler integrals of Eisenstein series in general, we recall basic facts about Eisenstein series in this section.

Throughout, we write $D = \frac{d}{d\tau}$ and $q = e^{2\pi i\tau}$. For $n \geq 0$, we denote with E_n the n th Euler number, defined by

$$(12) \quad \operatorname{sech} x = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!},$$

where $|x| < \pi$.

Lemma 4.1. *We have*

$$D^{2m}[\psi_{2m}(\tau/2)] = \frac{(2m)!}{\pi} \sum'_{k,j \in \mathbb{Z}} \frac{\chi_{-4}(j)}{(k\tau + j)^{2m+1}} - \frac{(-1)^m E_{2m} \pi^{2m}}{2^{2m+1}},$$

where $\chi_{-4} = \left(\frac{-4}{\cdot}\right)$ is the nonprincipal Dirichlet character modulo 4 (that is, $\chi_{-4}(n) = 0$ for even n , and $\chi_{-4}(n) = (-1)^{(n-1)/2}$ for odd n).

Proof. In light of the partial fraction expansion of the secant function

$$\sec\left(\frac{\pi\tau}{2}\right) = \frac{4}{\pi} \sum_{j \geq 1} \frac{\chi_{-4}(j)j}{j^2 - \tau^2} = \lim_{N \rightarrow \infty} \frac{2}{\pi} \sum_{j=-N}^N \frac{\chi_{-4}(j)}{\tau + j},$$

we derive that

$$D^k \sec\left(\frac{\pi\tau}{2}\right) = \frac{2(-1)^k k!}{\pi} \sum_{j \in \mathbb{Z}} \frac{\chi_{-4}(j)}{(\tau + j)^{k+1}}.$$

Consequently,

$$(13) \quad \begin{aligned} D^{2m} \sum_{n \geq 1} \frac{\sec\left(\frac{\pi n\tau}{2}\right)}{n^{2m}} &= \frac{2(2m)!}{\pi} \sum_{k \geq 1} \sum_{j \in \mathbb{Z}} \frac{\chi_{-4}(j)}{(k\tau + j)^{2m+1}} \\ &= \frac{(2m)!}{\pi} \sum'_{k,j \in \mathbb{Z}} \frac{\chi_{-4}(j)}{(k\tau + j)^{2m+1}} - \frac{2(2m)!}{\pi} L(\chi_{-4}, 2m+1), \end{aligned}$$

where

$$L(\chi_{-4}, s) = \sum_{n=1}^{\infty} \frac{\chi_{-4}(n)}{n^s} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s},$$

is the Dirichlet L -series attached to χ_{-4} , also known as the Dirichlet beta function. Using Euler's well-known evaluation [Ayo74]

$$(14) \quad L(\chi_{-4}, 2m+1) = \frac{1}{2} \frac{(-1)^m E_{2m}}{(2m)!} \left(\frac{\pi}{2}\right)^{2m+1}$$

in (13), we complete the proof of Lemma 4.1. \square

Example 4.2. In the case $m = 1$, we find that

$$D^2 \psi_2(\tau/2) = \frac{2}{\pi} \sum'_{k,j \in \mathbb{Z}} \frac{\chi_{-4}(j)}{(k\tau + j)^3} - \frac{\pi^2}{8},$$

since $L(\chi_{-4}, 3) = \frac{\pi^3}{32}$. The Eisenstein series

$$\sum'_{k,j \in \mathbb{Z}} \frac{\chi_{-4}(j)}{(4k\tau + j)^3} = \frac{\pi^3}{4} \left[\frac{1}{4} - q - q^2 + 8q^3 - q^4 - 26q^5 + \dots \right]$$

is a modular form of weight 3, level 4 and character χ_{-4} .

Define, as in [Miy06, Chapter 7], the Eisenstein series

$$(15) \quad E_k(\tau; \chi, \psi) = \sum'_{m,n \in \mathbb{Z}} \frac{\chi(m)\psi(n)}{(m\tau + n)^k},$$

where $k > 2$, and χ and ψ are Dirichlet characters modulo L and M , respectively. As detailed in [Miy06, Chapter 7], these Eisenstein series can be used to generate all Eisenstein series with respect to any congruence subgroup.

Example 4.3. By Lemma 4.1, the secant Dirichlet series ψ is connected with the case $\psi = \chi_{-4}$ and $\chi = 1$, the principal character modulo 1. To be precise,

$$(16) \quad D^{2m}[\psi_{2m}(\tau/2)] = \frac{(2m)!}{\pi} [E_{2m+1}(\tau; 1, \chi_{-4}) - E_{2m+1}(i\infty; 1, \chi_{-4})].$$

That the constant term on the right-hand side indeed agrees with the one stated in Lemma 4.1 will become clear from the facts about the Eisenstein series $E_k(\tau; \chi, \psi)$ which we review next.

Example 4.4. As detailed in Section 8.1, the cotangent Dirichlet series $\xi_s(\tau)$, introduced in (2), is in a similar way related to the Eisenstein series $E_{2m}(\tau; 1, 1)$, which is modular with respect to the full modular group.

In the sequel, we always assume that $\chi(-1)\psi(-1) = (-1)^k$ since, otherwise, $E_k(\tau; \chi, \psi) = 0$. In order to derive the Fourier expansion of the Eisenstein series, we recall that, by the character analogue of the Lipschitz summation formula [Ber75a], for any primitive Dirichlet character ψ of modulus M ,

$$\sum_{n=-\infty}^{\infty} \frac{\psi(n)}{(\tau + n)^s} = G(\psi) \frac{(-2\pi i/M)^s}{\Gamma(s)} \sum_{m=1}^{\infty} \bar{\psi}(m) m^{s-1} e^{2\pi i m \tau / M}.$$

Here, and in the sequel, $G(\psi) = \sum_{a=1}^M \psi(a)e^{2\pi ia/M}$ denotes the Gauss sum associated with ψ . If ψ is primitive, we thus find that

$$\begin{aligned}
E_k(\tau; \chi, \psi) &= a_0(E_k) + 2 \sum_{m=1}^{\infty} \chi(m) \sum_{n \in \mathbb{Z}} \frac{\psi(n)}{(m\tau + n)^k} \\
&= a_0(E_k) + 2G(\psi) \frac{(-2\pi i/M)^k}{\Gamma(k)} \sum_{m=1}^{\infty} \chi(m) \sum_{n=1}^{\infty} \bar{\psi}(n) n^{k-1} e^{2\pi i n m \tau / M} \\
(17) \quad &= a_0(E_k) + A \sum_{n=1}^{\infty} a_n(E_k) e^{2\pi i n \tau / M},
\end{aligned}$$

where $A = 2G(\psi)(-2\pi i/M)^k/\Gamma(k)$ and

$$a_0(E_k) = \begin{cases} 2L(k, \psi), & \text{if } \chi = 1, \\ 0, & \text{otherwise,} \end{cases} \quad a_n(E_k) = \sum_{d|n} \chi(n/d) \bar{\psi}(d) d^{k-1}.$$

We note that the series in (17) converges for any complex value of k , and hence (17) provides the analytic continuation of (15) to the entire complex k -plane.

Example 4.5. In light of Example 4.3, we therefore find that the q -expansion of the secant Dirichlet series is given by

$$(18) \quad \psi_{2m}(2\tau) = 2 \sum_{n \geq 1} \left(\sum_{d|n} \chi_{-4}(d) d^{2m} \right) \frac{q^n}{n^{2m}}.$$

Recall that the L -function of a modular form $f(\tau) = \sum_{n=0}^{\infty} b(n)e^{2\pi i n \tau / \lambda}$ is defined as

$$(19) \quad L(f, s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^{\infty} [f(i\tau) - f(i\infty)] \tau^{s-1} d\tau = \lambda^s \sum_{n=1}^{\infty} \frac{b(n)}{n^s}.$$

As another consequence of (17), if ψ is primitive, the L -function of $E(\tau) = E_k(\tau; \chi, \psi)$ is given by

$$(20) \quad L(E, s) = AM^s L(\chi, s) L(\bar{\psi}, 1 - k + s).$$

Depending on the parity of s , the values of $L(E, s)$ can be evaluated in terms of generalized Bernoulli numbers $B_{n, \chi}$, which are defined by

$$(21) \quad \sum_{n=0}^{\infty} B_{n, \chi} \frac{x^n}{n!} = \sum_{a=1}^L \frac{\chi(a) x e^{ax}}{e^{Lx} - 1},$$

if χ is a Dirichlet character modulo L . We observe that, for $n \neq 1$, the classical Bernoulli numbers B_n are equal to $B_{n, \chi}$ with $\chi = 1$. Similarly, the Euler numbers are connected with the case $\chi = \chi_{-4}$:

$$(22) \quad \frac{1}{2} \frac{E_{2m}}{(2m)!} = -\frac{B_{2m+1, \chi_{-4}}}{(2m+1)!}.$$

Generalized Bernoulli numbers are intimately related to values of Dirichlet L -series. Let $n > 0$ be an integer and χ a primitive Dirichlet character of conductor L such

that $\chi(-1) = (-1)^n$. Then, as detailed, for instance, in [Miy06, Thm. 3.3.4],

$$(23) \quad \begin{aligned} L(n, \chi) &= (-1)^{n-1} \frac{G(\chi)}{2} \left(\frac{2\pi i}{L} \right)^n \frac{B_{n, \bar{\chi}}}{n!}, \\ L(1-n, \chi) &= -B_{n, \chi}/n. \end{aligned}$$

On the other hand, if $\chi(-1) \neq (-1)^n$, then $L(1-n, \chi) = 0$ unless $\chi = 1$ and $n = 1$.

Finally, let us recall the basic transformation properties of the Eisenstein series $E_k(\tau; \chi, \psi)$, which are detailed, for instance, in [Miy06, Chapter 7]. Denote with $\Gamma_0(L, M)$ the group of matrices $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ such that $M|b$ and $L|c$. For any such γ ,

$$(24) \quad E_k(\tau; \chi, \psi)|_k \gamma = \chi(d) \bar{\psi}(d) E_k(\tau; \chi, \psi).$$

Moreover,

$$(25) \quad E_k(\tau; \chi, \psi)|_k S = \chi(-1) E_k(\tau; \psi, \chi).$$

5. REVIEW OF EICHLER INTEGRALS

In the language of Eichler integrals, to be reviewed next, the observation of Section 4 becomes the simple statement that the secant Dirichlet series ψ_{2m} are Eichler integrals of weight $2m+1$ Eisenstein series of level 2. As such, it is natural that the ψ_{2m} satisfy modular functional equations as in Section 2. In fact, it becomes *a priori* clear that the ψ_{2m} satisfy modular relations such as (3), in which the coefficients of the rational function on the right-hand side are determined by the period polynomial of the Eichler integral.

Lemma 4.1 shows that

$$F_{2m}(\tau) := \psi_{2m}(\tau) + \frac{E_{2m}}{2(2m)!} (\pi i \tau)^{2m}$$

has the property that its $2m$ th derivative is a modular form with respect to $\Gamma = \langle A, B \rangle$ of weight $2m+1$ (with A and B as defined in (8)). In other words, F_{2m} is an *Eichler integral* (we adopt the common custom and also refer to ψ_{2m} as an Eichler integral). In particular, for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

$$(26) \quad (c\tau + d)^{2m-1} F_{2m}(\gamma\tau) - F_{2m}(\tau)$$

is a polynomial of degree $2m-1$, the *period polynomial* of F_{2m} . Rewriting (26) in terms of $\psi_{2m}(\tau)$, we find that $\psi_{2m}(\tau)$ satisfies a functional equation of the form (3), where the rational function on the right-hand side is expressed in terms of the period polynomial of F_{2m} . For the general theory of period polynomials we refer to [PP13] and the references therein. A very brief introduction, suitable for our purposes, is given next.

Remark 5.1. A direct way to see that (26) is indeed a polynomial of degree $2m-1$ is offered by Bol's identity [Bol49]. It states that, for all sufficiently differentiable F and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$,

$$(27) \quad (D^{k+1}F)(\gamma\tau) = (c\tau + d)^{k+2} D^{k+1} [(c\tau + d)^k F(\gamma\tau)].$$

For the present purpose, we apply (27) with $F = F_{2m}$ and $k = 2m-1$.

In the following we adopt the notation of [PP13]. Let \mathcal{A} be the space of holomorphic functions on the upper half-plane \mathcal{H} . As usual, $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$ acts on \mathcal{H} by linear fractional transformations, and on \mathcal{A} via the slash operators; namely, if $f \in \mathcal{A}$ and k is an integer, then

$$(f|_k g)(\tau) = (c\tau + d)^{-k} f(g\tau), \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1.$$

This action extends naturally to the group algebra $\mathbb{C}[\Gamma_1]$.

As usual, we denote with T , S and R the matrices

$$(28) \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

and recall that the matrices T and S generate Γ_1 .

From now on, let f be a (not necessarily cuspidal) modular form of integral weight $k \geq 2$ with respect to Γ , with Γ being a subgroup of finite index of Γ_1 . Note that the modularity of f implies that $f|_k g = f$ for all $g \in \Gamma$. In the sequel, we will abbreviate $f|g = f|_k g$ since this is the only action of Γ_1 on modular forms of weight k that we consider.

Throughout this section, $w = k - 2$. Let V_w be the space of complex polynomials of degree at most w . The (multiple) period polynomial, introduced in [PP13], attached to f is the map $\rho_f : \Gamma \backslash \Gamma_1 \rightarrow V_w$ defined by

$$(29) \quad \rho_f(A)(X) = \int_0^{i\infty} [f|A(t) - a_0(f|A)] (t - X)^w dt.$$

In the sequel, we will often omit the dependence on X and just write $\rho_f(A)$ for the left-hand side. The goal of the final set of definitions is to connect these period polynomials, whose coefficients encode the critical L -values of f , to the transformation properties of Eichler integrals of f . The (multiple) Eichler integral of f , introduced in [PP13], is the function $\tilde{f} : \Gamma \backslash \Gamma_1 \rightarrow \mathcal{A}$ defined by

$$(30) \quad \tilde{f}(A)(\tau) = \int_\tau^{i\infty} [f|A(z) - a_0(f|A)] (z - \tau)^w dz,$$

with $a_0(f) = f(i\infty)$ denoting the constant term of the Fourier expansion of f . If $g \in \Gamma_1$ then $\tilde{f}|g(A) = \tilde{f}(Ag^{-1})|_{-w} g$ defines an action of Γ_1 , and hence $\mathbb{C}[\Gamma_1]$, on functions $\tilde{f} : \Gamma \backslash \Gamma_1 \rightarrow \mathcal{A}$.

The following result is [PP13, Proposition 8.1], which may also be found in [Wei77], where it is expressed in slightly different terms.

Proposition 5.2. *With f as above, define $\hat{\rho}_f = \tilde{f}|(1-S)$. Then, for any $A \in \Gamma \backslash \Gamma_1$,*

$$\hat{\rho}_f(A) = \rho_f(A) + (-1)^w \frac{a_0(f|A)}{w+1} X^{w+1} + \frac{a_0(f|AS^{-1})}{w+1} X^{-1}.$$

Note that matters simplify when only cusp forms are considered; in that case, ρ_f and $\hat{\rho}_f$ coincide. Though not necessarily a polynomial in the Eisenstein case, we will also refer to $\hat{\rho}_f$, as well as to $\tilde{f}|(1-\gamma)$ for $\gamma \in \Gamma_1$, as period polynomials of f .

Example 5.3. Let us make these definitions very explicit in the case where f is the Eisenstein series $E(\tau) = E_k(\tau; \chi, \psi)$ introduced in (15). Then $E|S = \chi(-1)E_k(\tau; \psi, \chi)$ by (25), and thus

$$(31) \quad \hat{\rho}_E(I) = \tilde{E}_k(X; \chi, \psi) - \psi(-1)X^{k-2}\tilde{E}_k(-1/X; \psi, \chi),$$

assuming that $\chi(-1)\psi(-1) = (-1)^k$ (since, otherwise, $E = 0$). Here, and in the sequel, we denote, with a slight abuse of notation,

$$\tilde{E}_k(\tau; \chi, \psi) = \tilde{E}(I)(\tau),$$

where the right-hand side is defined by (30). It follows from the Fourier expansion (17) that, for primitive ψ ,

$$(32) \quad \tilde{E}_k(\tau; \chi, \psi) = -\frac{4\pi i}{k-1} \frac{G(\psi)}{M} \sum_{n=1}^{\infty} \left(\sum_{d|n} \bar{\psi}(n/d) \chi(d) d^{1-k} \right) e^{2\pi i n \tau / M}.$$

Example 5.4. Following Example 4.3, we observe from (16) that

$$(33) \quad \psi_{2m}(\tau/2) = \frac{2m}{\pi} \tilde{E}_{2m+1}(\tau; 1, \chi_{-4}),$$

thus making explicit the nature of ψ_{2m} as an Eichler integral. In the present context, the functional equation (6), on replacing τ with $\tau/2$ and the appropriate scaling, translates into

$$(34) \quad \tilde{E}_{2m+1}(\tau; 1, \chi_{-4})|_{1-2m}(R-1) = \frac{\pi^{2m+1}}{2m} [z^{2m-1}] \frac{\sin(\tau z/2)}{\sin(z) \sin((\tau+1)z)},$$

where R is as in (28).

The next result allows us to express the left-hand side of (34) in terms of $\hat{\rho}_{E_{2m+1}}$ which, in the sense of Proposition 5.2, is a period polynomial of E_{2m+1} .

Proposition 5.5. *Let R be as in (28). Let f be a modular form for a group $\Gamma \leq \Gamma_1$, and let n be such that $R^n \in \Gamma$. Then*

$$\tilde{f}|(1-R^n)(I) = \hat{\rho}_f|(1-R^n)(I).$$

Proof. For any $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, it follows from the definition of $\hat{\rho}_f$ that

$$\hat{\rho}_f|(1-C) = \tilde{f}|(1-S)(1-C) = \tilde{f}|(1-C) - \tilde{f}|(S-SC).$$

It therefore suffices to show that $\tilde{f}|S(I) = \tilde{f}|SR^n(I)$. To see this, we observe that, for $C \in \Gamma$ as above,

$$\tilde{f}|SC(I) = \tilde{f}(S^{-1})|_{-w}SC = (a\tau + b)^w \int_{SC\tau}^{i\infty} [f|S^{-1}(z) - a_0(f|S^{-1})] (z - SC\tau)^w dz.$$

The change of variables $z = SCS^{-1}z'$ yields

$$\tilde{f}|SC(I) = \tau^w \int_{S\tau}^{a/b} [f|S^{-1}(z) - (-bz + a)^{-k} a_0(f|S^{-1})] (z - S\tau)^w dz,$$

and the desired equality follows because, upon setting $a = 1$ and $b = 0$, the right-hand side does not depend on the value of c . \square

6. PERIOD POLYNOMIALS OF EISENSTEIN SERIES

In the case $\chi = 1$ and $\psi = 1$, the Eisenstein series $E(\tau) = E_{2k}(\tau; 1, 1)$ is the usual Eisenstein series of weight $2k$ with respect to the full modular group. Its period polynomial, defined in Proposition 5.2 and made explicit in (31) for Eisenstein series, is well-known to be

$$(35) \quad \hat{\rho}_E(I) = -\frac{(2\pi i)^{2k}}{2k-1} \left[\sum_{s=0}^k \frac{B_{2s}}{(2s)!} \frac{B_{2k-2s}}{(2k-2s)!} X^{2k-2s-1} + \frac{\zeta(2k-1)}{(2\pi i)^{2k-1}} (X^{2k-2} - 1) \right];$$

compare, for instance, [Zag91, (11)]. On the other hand, this evaluation is equivalent to the formula (48), which Ramanujan famously recorded and which we briefly discuss in Section 8.1. A beautiful account of this connection is contained in [GMR11].

The (Laurent) polynomials on the right-hand side of (35) have interesting properties, which have been studied, for instance, in [GMR11], [MSW11], and [LR13]. In Section 7, we indicate that the generalized polynomials obtained in this section for higher level share similar properties.

In light of Example 5.4 and Proposition 5.5, the functional equations satisfied by the secant Dirichlet series are determined by the period polynomials associated to the Eisenstein series $E_{2k+1}(\tau; 1, \chi_{-4})$. We next compute the period polynomials of the Eisenstein series $E_{2k+1}(\tau; \chi, \psi)$, with χ and ψ being any pair of primitive Dirichlet characters.

Theorem 6.1. *Let $k \geq 3$, and let χ and ψ be primitive Dirichlet characters modulo L and M , respectively, such that $\chi(-1)\psi(-1) = (-1)^k$. For the Eisenstein series $E(\tau) = E_k(\tau; \chi, \psi)$, defined in (15),*

$$(36) \quad \begin{aligned} \hat{\rho}_E(I) &= -\frac{4\psi(-1)}{k-1} \sum_{\substack{s=0 \\ \chi(-1)=(-1)^s}}^k L(s, \chi)L(k-s, \psi)X^{k-s-1} \\ &\quad - \frac{2\psi(-1)}{k-1} \pi i [\varepsilon_\chi L(k-1, \psi)X^{k-2} - \varepsilon_\psi L(k-1, \chi)]. \end{aligned}$$

Here, $\varepsilon_\chi = 1$ if $\chi = 1$, and $\varepsilon_\chi = 0$ otherwise.

Proof. We first observe, from the definition (29), the general fact that, for a modular form E of weight k ,

$$\rho_E(A) = (-1)^{k-1} \sum_{s=1}^{k-1} \binom{k-2}{s-1} \frac{\Gamma(s)}{(2\pi i)^s} L(E|A, s)X^{k-s-1},$$

with the L -function of E as defined in (19). Let s be an integer with $0 < s < k$. We deduce from (20) and the functional equation, given, for instance, in [Apo76, Theorem 12.11, p. 263],

$$(37) \quad L(1-s, \bar{\psi}) = \frac{M^{s-1}\Gamma(s)}{(2\pi)^s} (e^{-\pi is/2} + \psi(-1)e^{\pi is/2})\tau(\bar{\psi})L(s, \psi)$$

that, for s such that $\chi(-1) = (-1)^s$,

$$L(E, s) = 4(-1)^s (2\pi i)^s \frac{\Gamma(k-s)}{\Gamma(k)} L(s, \chi)L(k-s, \psi).$$

On the other hand, for s such that $\chi(-1) \neq (-1)^s$, we have $L(\bar{\psi}, 1 - k + s) = 0$, which implies that $L(E, s) = 0$ unless $s = 1$, $\chi = 1$ or $s = k - 1$, $\psi = 1$. Combining these, we find that

$$\begin{aligned} \rho_E(I) &= -\frac{4\psi(-1)}{k-1} \sum_{\substack{s=1 \\ \chi(-1)=(-1)^s}}^{k-1} L(s, \chi)L(k-s, \psi)X^{k-s-1} \\ &\quad - \frac{(-1)^k}{2\pi i} \varepsilon_\chi L(E, 1)X^{k-2} - \frac{(-1)^k}{(2\pi i)^{k-1}} \Gamma(k-1) \varepsilon_\psi L(E, k-1), \end{aligned}$$

where the sum is over all integers s such that $0 < s < k$ and $\chi(-1) = (-1)^s$. From (20), together with the functional equation (37) of the involved Dirichlet L -series, we deduce that, assuming $\chi = 1$,

$$L(E, 1) = \frac{(2\pi i)^2}{k-1} L(k-1, \psi).$$

On the other hand, if $\psi = 1$, then

$$L(E, k-1) = -\frac{(-2\pi i)^k}{(k-1)!} L(k-1, \chi).$$

It follows from Proposition 5.2 that

$$\hat{\rho}_E(I) = \rho_E(I) + (-1)^k \frac{a_0(E)}{k-1} X^{k-1} + \frac{a_0(E|S^{-1})}{k-1} X^{-1}.$$

The values for $a_0(E)$ and $a_0(E|S^{-1}) = (-1)^k a_0(E|S)$ are given by (17) in combination with (25). Finally, using the fact that $L(0, \chi) = 0$ for any even Dirichlet character $\chi \neq 1$, we obtain (36). \square

Observe that, in the case $\chi = \psi = 1$, using Euler's identity [Ayo74]

$$\zeta(2m) = (-1)^{m+1} \frac{B_{2m}}{2(2m)!} (2\pi)^{2m},$$

we obtain from Theorem 6.1 the well-known special case (35). In the same spirit, Theorem 6.1 may always be rewritten, using (23), in terms of generalized Bernoulli numbers as we record below. For the cases where $\chi = 1$ or $\psi = 1$ the appropriate extra terms need to be inserted.

Corollary 6.2. *Under the assumptions of Theorem 6.1, if, additionally, χ and ψ are both nonprincipal, then*

$$\hat{\rho}_E(I) = -\chi(-1)G(\chi)G(\psi) \frac{(2\pi i)^k}{k-1} \sum_{s=0}^k \frac{B_{k-s, \bar{\chi}}}{(k-s)!L^{k-s}} \frac{B_{s, \bar{\psi}}}{s!M^s} X^{s-1}.$$

Remark 6.3. We note that similar results, based on residue calculations in the spirit of Section 2, are obtained in [Ber75b]. In fact, the Eisenstein series considered in [Ber75b], namely

$$\sum'_{m, n \in \mathbb{Z}} \frac{\chi(m)\psi(n)}{((m+r_1)\tau + n+r_2)^k},$$

have the two extra parameters r_1 and r_2 in comparison with $E_k(\tau; \chi, \psi)$. However, the analysis in [Ber75b] is restricted to the case when χ and ψ are primitive characters of the same modulus.

Example 6.4. In the case $\chi = \psi = 1$, Theorem 6.1 reduces to Ramanujan's identity (35), which in particular yields an interesting formula for the odd zeta values $\zeta(2k-1)$; see also [Gro70]. This formula has inspired much research, such as [GMR11]. Analogous formulas for other Dirichlet L -series have been derived in [Kat74] using partial fraction expansions. We will now illustrate how the results of [Kat74] follow from the special case $\psi = 1$ of Theorem 6.1, thus providing an alternative proof. In the setting of Theorem 6.1, with $\psi = 1$ and $\chi \neq 1$, we have

$$\hat{\rho}_E(I) = \frac{2\pi i}{k-1} L(k-1, \chi) - \frac{4}{k-1} \sum_{j=0}^{\lfloor k/2 \rfloor} L(k-2j, \chi) \zeta(2j) X^{2j-1}.$$

On the other hand, from (31),

$$\hat{\rho}_E(I) = \tilde{E}_k(\tau; \chi, 1) - \tau^{k-2} \tilde{E}_k(-1/\tau; 1, \chi).$$

Using the Fourier expansion (32), as well as $G(\chi)G(\bar{\chi}) = \chi(-1)L$ and the simple summations

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\sum_{d|n} \chi(d) d^{1-k} \right) q^n &= \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{k-1}} \frac{q^n}{1-q^n}, \\ \sum_{n=1}^{\infty} \left(\sum_{d|n} \chi(n/d) d^{1-k} \right) q^n &= \sum_{a=1}^L \chi(a) \sum_{n=1}^{\infty} \frac{1}{n^{k-1}} \frac{q^{an}}{1-q^{Ln}}, \end{aligned}$$

we obtain

$$\hat{\rho}_E(I) = \frac{4\pi i}{k-1} \left[F_1(\tau) - \frac{(-\tau)^{k-2}}{G(\bar{\chi})} F_2(-1/\tau) \right],$$

where, similar to [Kat74],

$$\begin{aligned} F_1(\tau) &= \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{k-1}} \frac{e^{2\pi i n \tau}}{e^{2\pi i n \tau} - 1}, \\ F_2(\tau) &= \sum_{a=1}^L \bar{\chi}(a) \sum_{n=1}^{\infty} \frac{1}{n^{k-1}} \frac{e^{2\pi i a n \tau / L}}{e^{2\pi i n \tau} - 1}. \end{aligned}$$

Solving for $L(k-1, \chi)$, we have arrived at

$$\frac{1}{2} L(k-1, \chi) = F_1(\tau) - \frac{(-\tau)^{k-2}}{G(\bar{\chi})} F_2(-1/\tau) + \frac{1}{\pi i} \sum_{j=0}^{\lfloor k/2 \rfloor} L(k-2j, \chi) \zeta(2j) \tau^{2j-1},$$

which is equivalent to the main result of [Kat74]. Note that this formula expresses the L -value as a combination of two Eichler integrals and a power of π times a Laurent polynomial with rational coefficients. In Example 7.7, similar formulas for these L -values are given, where only one Eichler integral is involved (evaluated at two arguments) and the polynomials appear to have the additional property of having all their nonreal roots on the unit circle.

With Theorem 6.1 in place, it is easy to deduce functional equations such as (6), which is the case $\chi = 1$, $\psi = \chi_{-4}$ of the next result. Note that the restriction to $\psi \neq 1$ is just to avoid the presence of an additional term. The case $\psi = 1$ is discussed in Example 7.7, where the formulas promised at the end of Example 6.4 are derived.

Corollary 6.5. *Let $k \geq 3$, and let χ and ψ be primitive Dirichlet characters modulo L and M , respectively, such that $\chi(-1)\psi(-1) = (-1)^k$. Let R be as in (28). If $\psi \neq 1$, then, for any integer n such that $L|n$,*

$$\tilde{E}_k(X; \chi, \psi)|_{2-k}(1-R^n) = -\frac{4\psi(-1)}{k-1} \sum_{\substack{s=0 \\ \chi(-1)=(-1)^s}}^k L(s, \chi)L(k-s, \psi)X^{k-s-1}(1-(nX+1)^{s-1}).$$

Proof. Recall from (24) that $E(\tau) = E_k(\tau; \chi, \psi)$ is modular with respect to $\Gamma_0(L, M)$. Since $R^n \in \Gamma_0(L, M)$, we may apply Proposition 5.5 and Theorem 6.1 to obtain

$$\tilde{E}_k(X; \chi, \psi)|_{2-k}(1-R^n) = \tilde{E}|(1-R^n)(I) = \hat{\rho}_E(I)|_{-w}(1-R^n),$$

with the Laurent polynomial $\hat{\rho}_E(I)$ given explicitly in (36). Note that

$$X^j|_{-w}(1-R^n) = X^j - (nX+1)^w \left(\frac{X}{nX+1} \right)^j = X^j \left(1 - (nX+1)^{w-j} \right).$$

In particular, $X^w|_{-w}B = X^w$, so that the term in (36) involving $L(k-1, \psi)$, if at all existent, is eliminated in $\tilde{E}|(1-R^n)(I)$. \square

Example 6.6. As indicated in Example 5.4, Corollary 6.5 specializes to a variation of Theorem 2.1 on setting $\chi = 1$ and $\psi = \chi_{-4}$. Namely, using (22) to relate the generalized Bernoulli numbers to Bernoulli and Euler numbers, we find that, for any positive integer m , the secant Dirichlet series satisfies the functional equation

$$(38) \quad \begin{aligned} & (2\tau+1)^{2m-1} \psi_{2m} \left(\frac{\tau}{2\tau+1} \right) - \psi_{2m}(\tau) \\ &= (\pi i)^{2m} \sum_{n=0}^m \frac{2^{2n-1} B_{2n} E_{2m-2n}}{(2n)!(2m-2n)!} \tau^{2m-2n} [1 - (2\tau+1)^{2n-1}]. \end{aligned}$$

We note that (38) is the same functional equation as (6) but representing the right-hand side in a somewhat different way; see Remark 6.7.

Remark 6.7. Let us indicate how to see, in a direct fashion, that (6) and (38) represent the same functional equation. Note that (38) can be expressed as

$$(39) \quad \psi_{2m}(\tau)|_{1-2m}(1-B) = \tau^{2m-1} [h_{2m}(\frac{1}{\tau}) - h_{2m}(2 + \frac{1}{\tau})],$$

where $h_{2m}(\tau)$ is the rational function

$$(40) \quad h_{2m}(\tau) = (\pi i)^{2m} \sum_{n=0}^m \frac{B_{2n} E_{2m-2n}}{(2n)!(2m-2n)!} (2\tau)^{2n-1}.$$

By the definitions of the Bernoulli and Euler numbers, (7) and (12), we find that

$$h_{2m}(\tau) = \frac{1}{2} \pi^{2m} [z^{2m-1}] \cot(\tau z) \sec(z).$$

Hence, the right-hand side of (38) and (39) equals

$$\frac{1}{2} \pi^{2m} [z^{2m-1}] \sec(\tau z) [\cot(z) - \cot((2\tau+1)z)].$$

The equivalence of (6) and (38) then follows from

$$\sec(\tau z) [\cot(z) - \cot((2\tau+1)z)] = \frac{\sin(\tau z)}{\sin(z) \sin((2\tau+1)z)},$$

which is obtained from basic trigonometric identities.

As an application of Corollary 6.5, we now derive formulas for the values of ψ_{2m} at families of real quadratic irrationalities, thus complementing and illustrating the results of Section 3 in an explicit fashion.

Example 6.8. Since Corollary 6.5 applies to all powers of R , which lie in the appropriate modular subgroup, we find that, for positive integers m and integers μ ,

$$(41) \quad \begin{aligned} & (2\mu\tau + 1)^{2m-1} \psi_{2m} \left(\frac{\tau}{2\mu\tau + 1} \right) - \psi_{2m}(\tau) \\ &= (\pi i)^{2m} \sum_{n=0}^m \frac{2^{2n-1} B_{2n} E_{2m-2n}}{(2n)!(2m-2n)!} \tau^{2m-2n} [1 - (2\mu\tau + 1)^{2n-1}]. \end{aligned}$$

We now demonstrate how to use these functional equations to obtain families of explicit evaluations of ψ_{2m} at certain real quadratic irrationalities. Let τ_0 be fixed by $A^\lambda B^\mu A^\nu$, that is $A^\lambda B^\mu A^\nu \tau_0 = \tau_0$. A brief calculation shows that

$$\tau_0 = \lambda - \nu \pm \sqrt{(\lambda + \nu) \left(\frac{1}{\mu} + (\lambda + \nu) \right)}.$$

Denote with $T_{m,\mu} = \psi_{2m}|_{1-2m}(B^\mu - I)$ the right-hand side of (41). It follows from

$$\psi_{2m}(A^\lambda B^\mu A^\nu \tau) = \frac{\psi_{2m}(A^\nu \tau) + T_{m,\mu}(A^\nu \tau)}{(2\mu A^\nu \tau + 1)^{2m-1}},$$

together with the fact that τ_0 is fixed by $A^\lambda B^\mu A^\nu$, that

$$\psi_{2m}(\tau_0) = \frac{T_{m,\mu}(\tau_0 + 2\nu)}{(2\mu(\tau_0 + 2\nu) + 1)^{2m-1} - 1}.$$

A straightforward, but slightly tedious, calculation using (41) and the explicit value of τ_0 shows that

$$(42) \quad \psi_{2m}(\tau_0) = -(\pi i)^{2m} \sum_{n=0}^m \frac{2^{2n-1}}{\mu^{2m-2n}} \frac{B_{2n} E_{2m-2n}}{(2n)!(2m-2n)!} \frac{r_n}{r_m},$$

where r_n are the rational numbers

$$r_n = \frac{1}{2} [(1 + \sqrt{\alpha})^{2n-1} + (1 - \sqrt{\alpha})^{2n-1}] = \begin{cases} \frac{1}{1-\alpha}, & \text{if } n = 0, \\ \sum_{j=0}^{n-1} \binom{2n-1}{2j} \alpha^j, & \text{if } n \geq 1, \end{cases}$$

and $\alpha = \frac{1}{\mu(\nu+\lambda)} + 1$. Note that this is an explicit illustration of the general fact, proved in Theorem 3.4, that $\psi_{2m}(\tau)$ is a rational multiple of π^{2m} whenever τ is a real quadratic irrationality.

We note that the right-hand side of (42) only depends on $\kappa = \nu + \lambda$ but not on ν and λ individually. For the left-hand side, this follows from the obvious periodicity relation $\psi_{2m}(\tau + 2) = \psi_{2m}(\tau)$. The first two cases of (42) can thus be stated, in equivalent forms, as

$$\begin{aligned} \psi_2 \left(\kappa + \sqrt{\kappa \left(\frac{1}{\mu} + \kappa \right)} \right) &= \frac{\pi^2}{6} \left(1 + \frac{3\kappa}{2\mu} \right), \\ \psi_4 \left(\kappa + \sqrt{\kappa \left(\frac{1}{\mu} + \kappa \right)} \right) &= \frac{\pi^4}{90} \left(1 + \frac{5\kappa}{2\mu} - \frac{5\kappa^2(16\mu^2 - 15)}{8\mu^2(4\kappa\mu + 3)} \right), \end{aligned}$$

where κ and μ are integers and $\mu \neq 0$. The first of these, in the special case $\kappa = 2\lambda$, is also given in [LRR14].

Remark 6.9. Using the methods of [Ber73], [Ber75b] or, alternatively, [Raz77], one can derive the general transformation laws of the Eichler integrals $\tilde{E}_k(\tau; \chi, \psi)$, and, in particular, $\psi_{2m}(\tau)$, under arbitrary elements of the full modular group. Here, we do not, however, pursue this further.

7. ZEROS OF GENERALIZED RAMANUJAN POLYNOMIALS

It has recently been shown in [CFI12] and [EGR13] that the nontrivial zeros of period polynomials of modular forms, which are Hecke eigenforms of level 1, all lie on the unit circle. In this section, we consider the Eisenstein case of higher level by investigating the zeros of the period polynomials calculated in the previous section. We again find that, at least conjecturally, most of the roots lie on a circle in the complex plane. The observations suggest that the problem solved by [CFI12] and [EGR13] is interesting in the higher level case as well.

An application of these considerations is that knowledge of the location of the zeros of the period polynomials calculated in the previous section gives rise to explicit formulas for Dirichlet L -values of “wrong” parity (that is, values at integers of parity opposite to the Dirichlet character) in terms of Eichler integrals. This is made explicit in Example 7.7. The special case of the principal character is detailed in [GMR11], in which case odd zeta values are expressed in terms of the difference of two Eichler integrals at algebraic argument of modulus 1.

For positive integer k , and Dirichlet characters χ and ψ modulo L and M , we define the *generalized Ramanujan polynomial*

$$(43) \quad R_k(X; \chi, \psi) = \sum_{s=0}^k \frac{B_{s,\chi}}{s!} \frac{B_{k-s,\psi}}{(k-s)!} \left(\frac{X-1}{M} \right)^{k-s-1} (1 - X^{s-1}).$$

Note that this is a polynomial if χ and ψ are both nonprincipal, and a Laurent polynomial otherwise. Further note that, if $\chi(-1)\psi(-1) \neq (-1)^k$, then $R_k(X; \chi, \psi) = 0$, unless $\psi = 1$ in which case

$$R_k(X; \chi, \psi) = \frac{1}{2} \frac{B_{k-1,\chi}}{(k-1)!} (1 - X^{k-2}).$$

In the sequel, we will therefore often assume, without loss of generality and as we did in previous sections, that $\chi(-1)\psi(-1) = (-1)^k$.

In [MSW11] the *Ramanujan polynomials* are, essentially, defined as

$$(44) \quad R_k(X) = \sum_{s=0}^k \frac{B_s}{s!} \frac{B_{k-s}}{(k-s)!} X^{s-1},$$

where k is an even integer (in [MSW11] the index k is shifted by 1, X appears with exponent s , and the definition differs for $k = 2$). The next result shows that the generalized Ramanujan polynomials, despite their different appearance, reduce to the Ramanujan polynomials when $\chi = 1$ and $\psi = 1$.

Proposition 7.1. *For $k > 1$, $R_{2k}(X; 1, 1) = R_{2k}(X)$.*

Proof. As evidenced by (35), the polynomial $R_{2k}(X)$ is the odd part of the period polynomial of the level 1 Eisenstein series $E_{2k}(\tau; 1, 1)$. As such it satisfies the relations [Zag91]

$$(45) \quad R_{2k}(X)|_{2-2k}(1+S) = R_{2k}(X)|_{2-2k}(1+U+U^2) = 0.$$

Here, $U = TS$ with T and S as defined in (28). On the other hand, by construction (43),

$$R_{2k}(X; 1, 1) = R_{2k}(X)|_{2-2k}(1-R)T^{-1}.$$

A brief calculation reveals that $(1+U+U^2)T = T+R+S$. Hence, using both relations (45), we find

$$R_{2k}(X)|_{2-2k}(1-R) = R_{2k}(X)|_{2-2k}(1+T+S) = R_{2k}(X)|_{2-2k}T,$$

which proves the claim. \square

The next example indicates that the definition (43) of the generalized Ramanujan polynomials is natural, by connecting them to period polynomials of generalized Eisenstein series studied in Section 6.

Example 7.2. As in Corollary 6.5, let $k \geq 3$, and let χ and $\psi \neq 1$ be primitive Dirichlet characters modulo L and M , respectively, such that $\chi(-1)\psi(-1) = (-1)^k$. Then, with R as in (28),

$$\tilde{E}_k(X; \chi, \psi)|_{2-k}(1-R^L) = -\chi(-1)G(\chi)G(\psi) \frac{(2\pi i/L)^k}{k-1} \frac{L}{M} R_k(LX+1; \bar{\chi}, \bar{\psi}).$$

In other words, up to some scaling and a linear change of variables, the polynomial $R_k(X; \bar{\chi}, \bar{\psi})$ is a period polynomial of the Eisenstein series $E_k(\tau; \chi, \psi)$.

Conjecture 7.3. *For nonprincipal real Dirichlet characters χ and ψ , the polynomial $R_k(X; \chi, \psi)$ is unimodular, that is, all its roots lie on the unit circle.*

We have verified Conjecture 7.3 numerically for all $k \leq 50$ and all characters of modulus up to 100. We note that it follows from Proposition 7.1 and the results in [MSW11] that all nonreal zeros of $R_k(X; 1, 1)$ lie on the unit circle. On the other hand, it is conjectured (in equivalent form) in [LRR14] that all roots of $R_k(X; 1, \chi_{-4})$ lie on the unit circle. Before giving further evidence in support of Conjecture 7.3 as well as an application, we indicate the conjectural situation in the cases $\chi = 1$ or $\psi = 1$, which is not included above. In vague summary, it appears that still most of the roots lie on the unit circle.

Example 7.4. Let χ be a nonprincipal real Dirichlet character. Computations show that, at least for $k \leq 50$ and χ of modulus at most 100, the polynomials $R_k(X; \chi, 1)$ are unimodular, except when χ takes the same values as χ_3 , the unique character of conductor 3. In the case $\chi = 1$, it was shown in [MSW11] that $R_{2k}(X; 1, 1)$, for $k \geq 2$, has exactly four distinct real roots (approaching $\pm 2^{\pm 1}$) and that the remaining roots lie on the unit circle. On the other hand, it appears that $R_{2k+1}(X; \chi_3, 1)$, for $k \geq 3$, has exactly three distinct real roots (-1 as well as two roots approaching $-2^{\pm 1}$) and that the remaining roots again lie on the unit circle.

In the next example, we restrict to primitive characters for expositional reasons and make use of the fact [ZR76] that, for a given conductor M , there is at most one primitive real Dirichlet character modulo M of each parity. We label even such

characters as $M+$ and odd ones as $M-$. For instance, the label $8+$ refers to the even real Dirichlet character of conductor 8.

Example 7.5. The situation in the case of $R_k(X; 1, \psi)$, with ψ a real primitive character, is slightly more varied. For certain characters ψ , such as

$$3-, 4-, 5+, 8\pm, 11-, 12+, 13+, 19-, 21+, 24+, \dots,$$

the polynomial $R_k(X; 1, \psi)$ appears to again be unimodular. For certain other characters ψ , such as

$$1+, 7-, 15-, 17+, 20-, 23-, 24-, \dots,$$

we observe, at least for small k , that all nonreal roots lie on the unit circle. On the other hand, there remains a third group of exceptional characters ψ , namely $35-, 59-, 83-, 131-, 155-, 179-, \dots$ (we observe that in each listed case ψ is odd), for which $R_k(X; 1, \psi)$ can have nonreal zeros off the unit circle. Consider, for instance, the unique real primitive Dirichlet character χ_{35} of conductor 35. Then the polynomial $R_7(X; 1, \chi_{35})$ has the seven roots (given to three decimal digits)

$$1, \quad 0.461 \pm 0.888i, \quad (-0.657 \pm 0.922i)^{\pm 1}.$$

While the first three listed roots have absolute value 1, the last four have absolute value $1.132^{\pm 1}$. In each of the exceptional cases, we observed, as in the example of $R_7(X; 1, \chi_{35})$, at most four nonreal zeros off the unit circle (in light of Proposition 7.6, such zeros necessarily come in groups of four).

In order for all zeros of a polynomial $p(X) = a_0 + a_1X + \dots + a_nX^n$, $a_n \neq 0$, to lie on the unit circle, it is a necessary condition [Coh22], [LS13], that the polynomial is self-inversive, that is, for some ε with $|\varepsilon| = 1$, $a_k = \varepsilon \overline{a_{n-k}}$ for $k = 0, 1, \dots, n$. In support of Conjecture 7.3, we now observe that, for real characters, $R_k(X; \chi, \psi)$ is self-inverse with $\varepsilon = \pm 1$. In other words, $R_k(X; \chi, \psi)$ is reciprocal or anti-reciprocal depending on the parity of ψ .

Proposition 7.6. *Let χ and ψ be real Dirichlet characters. If $\chi(-1)\psi(-1) = (-1)^k$, then*

$$R_k(X; \chi, \psi) = \psi(-1)X^{k-2}R_k(X^{-1}; \chi, \psi).$$

Proof. Temporarily, denote with

$$p_s(X) = (X-1)^{k-s-1}(1-X^{s-1})$$

one of the terms in (43) contributing to $R_k(X; \chi, \psi)$. It is simple to check that

$$X^{k-2}p_s(X^{-1}) = (-1)^{k-s}p_s(X).$$

On the other hand, recall that, for any Dirichlet character χ , $B_{s,\chi} = 0$ if $\chi(-1) \neq (-1)^s$, unless $\chi = 1$ and $s = 1$. It follows that $B_{k-s,\psi} = 0$ if $\psi(-1) \neq (-1)^{k-s}$, unless $\psi = 1$ and $s = k-1$. In the latter case, when $\psi = 1$ and $s = k-1$, we have $B_{s,\chi} = B_{k-1,\chi} = 0$ because $\chi(-1) = (-1)^k$, unless $\chi = 1$ and $k = 2$, which may be checked separately. We have thus shown that $X^{k-2}p_s(X^{-1}) = \psi(-1)p_s(X)$ for all s that have a nonzero coefficient in (43). \square

Example 7.7. The case $\psi = 1$ is of special interest, because it yields explicit formulas for Dirichlet L -values at integral arguments (of parity opposite to the Dirichlet character) in terms of Eichler integrals. Let χ be a primitive Dirichlet

character and $k \geq 3$ such that $\chi(-1) = (-1)^k$. Applying Theorem 6.1 as in Corollary 6.5 yields

$$\begin{aligned} \frac{k-1}{2\pi i} \tilde{E}_k(X; \chi, 1)|_{2-k}(1-R^L) &= G(\chi) (-2\pi i/L)^{k-1} R_k(LX+1; \bar{\chi}, 1) \\ &\quad + L(k-1, \chi) \left(1 - (LX+1)^{k-2}\right). \end{aligned}$$

Solving for $L(k-1, \chi)$, we obtain formulas for these L -values in the spirit of [Kat74]; see Example 6.4. On the other hand, suppose that α , with $\text{Im}(\alpha) > 0$, is a root of $R_k(\alpha; \bar{\chi}, 1) = 0$, which is not a $(k-2)$ th root of unity. Then

$$(46) \quad L(k-1, \chi) = \frac{k-1}{2\pi i(1-\alpha^{k-2})} \left[\tilde{E}_k\left(\frac{\alpha-1}{L}; \chi, 1\right) - \alpha^{k-2} \tilde{E}_k\left(\frac{1-1/\alpha}{L}; \chi, 1\right) \right],$$

thus explicitly linking the L -value to values of the Eichler integral at algebraic points, as is studied for $\chi = 1$ in [GMR11]. Note that by (32), as in Example 6.4,

$$\tilde{E}_k(\tau; \chi, 1) = \frac{4\pi i}{k-1} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{k-1}} \frac{e^{2\pi i n \tau}}{e^{2\pi i n \tau} - 1}.$$

Hence (46) takes the entirely explicit form

$$L(k-1, \chi) = \frac{2}{1-\alpha^{k-2}} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{k-1}} \left[\frac{1}{1-e^{2\pi i n(1-\alpha)/L}} - \frac{\alpha^{k-2}}{1-e^{2\pi i n(1/\alpha-1)/L}} \right],$$

which expresses the L -value as a combination of two Lambert-type series. The special case $\chi = 1$, which is a consequence of Ramanujan's identity (35), has recently been studied in [GMR11]. Specifically, using the notation

$$F_k(z) := \frac{ki}{4\pi} \tilde{E}_{k+1}(z; 1, 1) = \sum_{n=1}^{\infty} \frac{1}{n^k} \frac{e^{2\pi i n \tau}}{1 - e^{2\pi i n \tau}},$$

it is shown in [GMR11, Theorem 1.1] that the numbers

$$F_{2k+1}(\beta) - \beta^{2k} F_{2k+1}(-1/\beta) = \frac{(2k+1)i}{4\pi} \left[\tilde{E}_{2k+2}(\beta; 1, 1) - \beta^{2k} \tilde{E}_{2k+2}(-1/\beta; 1, 1) \right]$$

are transcendental for every algebraic $\beta \in \mathcal{H}$ with at most $2k+2+\delta$ exceptions. Here $\delta = 0, 1, 2, 3$ according to whether $\text{gcd}(k, 6) = 1, 2, 3$ or 6 , respectively. Examples of such exceptional values are $\beta = i$, if k is even, and $\beta = e^{\pi i/3}, e^{2\pi i/3}$, if k is divisible by 3 . In each of these three cases, however, the combination $F_{2k+1}(\beta) - \beta^{2k} F_{2k+1}(-1/\beta)$ vanishes. As pointed out in [GMR11], the existence of further exceptional β is very unlikely and would imply that $\zeta(2k+1)$ is an algebraic linear combination of 1 and π^{2k+1} . This conclusion relies on the results of [MSW11], by which the only roots of unity that are zeros of the corresponding period polynomial are $\pm i$, if k is even, and $\pm e^{\pi i/3}, \pm e^{2\pi i/3}$, if k is divisible by 3 . The above results indicate that similar transcendence results should hold for the more general Eichler integrals $\tilde{E}_k(\tau; \chi, 1)$ in place of $\tilde{E}_k(\tau; 1, 1)$, in which case the arithmetic nature of Dirichlet L -values of “wrong” parity is concerned.

Remark 7.8. It appears that the observations in this section can be extended further. For instance, in the case of imaginary Dirichlet characters, one finds that most of the zeros lie on the unit circle, if one considers instead of $R_k(X; \chi, \psi)$ its real or imaginary part (that is, the polynomial with coefficients which are the real

or imaginary parts of the coefficients of $R_k(X; \chi, \psi)$. Finally, generalizing (43), one may consider the polynomial

$$\sum_{s=0}^k \frac{B_{s,\chi}}{s!} \frac{B_{k-s,\psi}}{(k-s)!} \left(\frac{X-1}{m} \right)^{k-s-1} (1-X^{s-1}),$$

where $m = aM$ for some integer a . In the spirit of Example 7.2, this polynomial corresponds to the period polynomial in Corollary 6.5 when $n = aL$. It again appears that most roots of these polynomials lie on the unit circle.

As a second, and possibly more direct, generalization of the Ramanujan polynomials (44), we briefly also consider

$$(47) \quad S_k(X; \chi, \psi) = \sum_{s=0}^k \frac{B_{s,\chi}}{s!} \frac{B_{k-s,\psi}}{(k-s)!} \left(\frac{LX}{M} \right)^{k-s-1},$$

where χ and ψ are Dirichlet characters modulo L and M . The two generalizations (43) and (47) are related by

$$S_k(X; \chi, \psi)|_{2-k}(1-R^L) = R_k(LX+1; \chi, \psi).$$

By (31) and Corollary 6.2, we have that, for $k \geq 3$ and primitive nonprincipal Dirichlet characters χ and ψ such that $\chi(-1)\psi(-1) = (-1)^k$,

$$\tilde{E}_k(X; \chi, \psi) - \psi(-1)X^{k-2}\tilde{E}_k(-1/X; \psi, \chi) = -\chi(-1)G(\chi)G(\psi) \frac{(2\pi i/L)^k}{k-1} S_k(X; \bar{\chi}, \bar{\psi}),$$

which expresses the $S_k(X; \chi, \psi)$ as period polynomials as well. In general, these polynomials are not self-inverse and therefore cannot be unimodular.

Conjecture 7.9. *For nonprincipal real Dirichlet characters χ , all nonzero roots of the polynomial $S_k(X; \chi, \chi)$ lie on the unit circle.*

We have verified this conjecture numerically for $k \leq 50$ and characters χ of modulus at most 100. In the case $\chi = 1$, we recall that it was shown in [MSW11] that the polynomials $S_{2k}(X; 1, 1) = R_{2k}(X)$ have all their nonreal zeroes on the unit circle. On the other hand, for $\chi = 1$, the polynomial $S_{2k}(X; 1, 1)$ is only the odd part of the period polynomial and it was recently proved in [LS13] that the full period polynomial (35) is indeed unimodular.

8. RELATION TO SUMS CONSIDERED BY RAMANUJAN

8.1. **Sums of level 1.** Ramanujan famously recorded (see [Ber77] or [Ber89, p. 276], as well as the references therein) the formula

$$(48) \quad \alpha^{-m} \left\{ \frac{\zeta(2m+1)}{2} + \sum_{n=1}^{\infty} \frac{n^{-2m-1}}{e^{2\alpha n} - 1} \right\} = (-\beta)^{-m} \left\{ \frac{\zeta(2m+1)}{2} + \sum_{n=1}^{\infty} \frac{n^{-2m-1}}{e^{2\beta n} - 1} \right\} - 2^{2m} \sum_{n=0}^{m+1} (-1)^n \frac{B_{2n}}{(2n)!} \frac{B_{2m-2n+2}}{(2m-2n+2)!} \alpha^{m-n+1} \beta^n,$$

where α and β are positive numbers with $\alpha\beta = \pi^2$ and m is any nonzero integer. Rewriting equation (48) using

$$\frac{1}{e^x - 1} = \frac{1}{2} \cot\left(\frac{x}{2}\right) - \frac{1}{2},$$

slightly shifting the value of m and setting $\beta = \pi i\tau$, and therefore $\alpha = -\pi i/\tau$, we obtain, as in [Ber76], for integers $m \neq 1$,

$$\tau^{2m-2}\xi_{2m-1}\left(-\frac{1}{\tau}\right) - \xi_{2m-1}(\tau) = (-1)^m(2\pi)^{2m-1} \sum_{n=0}^m \frac{B_{2n}}{(2n)!} \frac{B_{2m-2n}}{(2m-2n)!} \tau^{2n-1},$$

where ξ_s is the cotangent Dirichlet series defined in (2).

As in the case of the secant Dirichlet series, discussed in Sections 4 and 6, the cotangent Dirichlet series is essentially an Eichler integral. Indeed, one easily checks that, for integral s ,

$$\frac{i}{2}\xi_s(\tau) = \frac{1}{2}\zeta(s) + \sum_{n=1}^{\infty} \frac{\sigma_s(n)}{n^s} q^n.$$

When $s = 2k - 1$ is odd, the right-hand side visibly is, up to scaling and a constant term, an Eichler integral of the weight $2k$ Eisenstein series

$$E_{2k}(\tau; 1, 1) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{\Gamma(2k)} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n,$$

which is modular with respect to the full modular group.

Remark 8.1. Proceeding as in Section 3, though matters simplify because the level is 1, we conclude that $\xi_{2m+1}(\sqrt{r})$ is a rational multiple of $\pi^{2m+1}\sqrt{r}$ whenever r is a positive rational number (assuming, for convergence, that \sqrt{r} is irrational). Explicit special cases of this observation may be found, for instance, in [Ber76].

8.2. Sums of level 4. Ramanujan also found [Ber89, Entry 21(iii), p. 277] the identity

$$(49) \quad \alpha^{-m+1/2} \left\{ \frac{1}{2}L(\chi_{-4}, 2m) + \sum_{n=1}^{\infty} \frac{\chi_{-4}(n)}{n^{2m}(e^{\alpha n} - 1)} \right\} = \frac{(-1)^m \beta^{-m+1/2}}{2^{2m+1}} \sum_{n=1}^{\infty} \frac{\operatorname{sech}(\beta n)}{n^{2m}} \\ + \frac{1}{4} \sum_{n=0}^m \frac{(-1)^n}{2^{2n}} \frac{E_{2n}}{(2n)!} \frac{B_{2m-2n}}{(2m-2n)!} \alpha^{m-n} \beta^{n+1/2},$$

which was first proved in print by Chowla [Cho28], where α and β are positive numbers with $\alpha\beta = \pi^2$ and m is any integer. The goal of this section is to relate (49) to the present discussion of the modular properties of the secant Dirichlet series. It will transpire that (49) is an explicit version of Theorem 6.1 in the case $\chi = 1$ and $\psi = \chi_{-4}$. In other words, equation (49) encodes how ψ_{2m} transforms under S , as defined in (28). From here, we can then work out the exact way in which ψ_{2m} transforms under any transformation of $\mathrm{SL}_2(\mathbb{Z})$, though we do not develop the details here (but see Remark 8.2 for indications). On the other hand, we demonstrate that we may use (49) as the basis for yet another derivation of the functional equation (6).

With $\alpha = \pi i\tau$, and proceeding as in the case of (48), we can write (49) as

$$(50) \quad \tau^{2m-1}\psi_{2m}\left(-\frac{1}{\tau}\right) = \hat{\psi}_{2m}(\tau) - h_{2m}(\tau),$$

where

$$\hat{\psi}_{2m}(\tau) = \sum_{n=1}^{\infty} \chi_{-4}(n) \frac{\cot\left(\frac{\pi n}{2}\tau\right)}{(n/2)^{2m}}$$

and $h_{2m}(\tau)$ is the rational function defined in (40). Since $\hat{\psi}_{2m}(\tau+2) = \hat{\psi}_{2m}(\tau)$, we find that

$$\psi_{2m}|_{1-2m}ST^{-2} = \hat{\psi}_{2m} - h_{2m}|_{1-2m}T^{-2}.$$

Using further that $B = ST^{-2}S^{-1}$, with B as in (8), we find that

$$(51) \quad \begin{aligned} \psi_{2m}|_{1-2m}B &= \psi_{2m} - h_{2m}|_{1-2m}(S^{-1} + T^{-2}S^{-1}) \\ &= \psi_{2m} + h_{2m}|_{1-2m}(S + T^{-2}S). \end{aligned}$$

which, in expanded form, is precisely the functional equation (38).

Remark 8.2. The functional equation satisfied by ψ_{2m} is given in [LRR14] in somewhat different form. Indeed, comparing the rational functions involved in the functional equations, we observe that

$$(52) \quad \begin{aligned} & (\pi i)^{2m} \sum_{n=0}^m \frac{2^{2n-1} B_{2n} E_{2m-2n}}{(2n)!(2m-2n)!} \tau^{2m-2n} [1 - (2\tau+1)^{2n-1}] \\ &= (\pi i)^{2m} \sum_{n=0}^m \frac{2^{2n-1} B_{2n}(\frac{1}{2}) E_{2m-2n}}{(2n)!(2m-2n)!} (\tau+1)^{2m-2n} [1 - (2\tau+1)^{2n-1}], \end{aligned}$$

where the first sum is the right-hand side of (38) and the second sum is, up to notation, the one derived in [LRR14]. To see that these two rational functions indeed coincide, one may proceed as in Remark 6.7. On the other hand, in order to further illustrate how ψ_{2m} transforms under the full modular group, we now sketch a proof of their equivalence, which, ultimately, derives from $B = ST^{-2}S^{-1} = (TST)^2$.

We can easily check that $\psi_{2m}|_{1-2m}T$ is given by

$$\psi_{2m}(\tau+1) = \sum_{n=1}^{\infty} (-1)^n \frac{\sec(\pi n \tau)}{n^{2m}} = \frac{1}{2^{2m-1}} \psi_{2m}(2\tau) - \psi_{2m}(\tau).$$

It follows that

$$\psi_{2m}|_{1-2m}TS = \hat{\psi}_{2m}(\tau/2) - \hat{\psi}_{2m}(\tau) - h_{2m}(\tau/2) + h_{2m}(\tau).$$

Since $\cot(z/2) - \cot(z) = \csc(z)$, this equals

$$\psi_{2m}|_{1-2m}TS = \sum_{n=1}^{\infty} \chi_{-4}(n) \frac{\csc(\frac{\pi n}{2}\tau)}{(n/2)^{2m}} - h_{2m}(\tau/2) + h_{2m}(\tau),$$

which then implies that

$$\begin{aligned} \psi_{2m}|_{1-2m}TST^2 &= - \sum_{n=1}^{\infty} \chi_{-4}(n) \frac{\csc(\frac{\pi n}{2}\tau)}{(n/2)^{2m}} - h_{2m}(\tau/2+1) + h_{2m}(\tau+2) \\ &= -\psi_{2m}|_{1-2m}TS + [h_{2m}(\tau) - h_{2m}(\tau/2)]|_{1-2m}(1+T^2). \end{aligned}$$

Since $B = (TST)^2 = (TST^2)ST$, we thus arrive at

$$(53) \quad \psi_{2m}|_{1-2m}B = \psi_{2m} + [h_{2m}(\tau) - h_{2m}(\tau/2)]|_{1-2m}(ST + T^2ST).$$

Comparing (53) with (51), we have shown that

$$h_{2m}(\tau)|_{1-2m}(S + T^{-2}S) = [h_{2m}(\tau) - h_{2m}(\tau/2)]|_{1-2m}(ST + T^2ST),$$

which, upon expanding and using the relation $B_n(\frac{1}{2}) = -(1 - 2^{1-n})B_n$, results in the desired equality (52).

Having thus come full circle, we close with remarking that a number of further infinite sums, similar in shape to (1) and (2), with trigonometric summands and modular properties are discussed in [Ber89, Chapter 14] and the references therein.

9. CONCLUSION

We have reviewed the well-known fact that, among similar sums, the cotangent and secant Dirichlet series of appropriate parity are Eichler integrals of Eisenstein series of level 1 and 4, respectively. Their functional equations, recorded by Ramanujan in his notebooks, are thus instances of the modular transformation properties of Eichler integrals in general, with their precise form determined by the corresponding period polynomials. This has led us to explicitly compute period polynomials of Eisenstein series of higher level. Motivated by recent results [GMR11], [MSW11], [LR13], [LS13] on the zeros of Ramanujan polynomials, which arise in the level 1 case, we observe that the generalized Ramanujan polynomials appear to also be (nearly) unimodular. On the other hand, it was recently shown in [CFI12] and [EGR13] that the nontrivial zeros of period polynomials of modular forms, which are Hecke eigenforms of level 1, all lie on the unit circle. Our observations for Eisenstein series of higher level suggest that it could be interesting to extend the results in [CFI12], [EGR13] to the case of higher level.

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REFERENCES

- [Apo76] T. Apostol. *Introduction to Analytic Number Theory*. Springer-Verlag, New York, 1976.
- [Ayo74] R. Ayoub. Euler and the zeta function. *Amer. Math. Monthly*, 81:1067–1086, 1974.
- [Ber73] B. C. Berndt. Character transformation formulae similar to those for the Dedekind eta-function. In H. Diamond, editor, *Proceedings of Symposia in Pure Mathematics*, volume 24, pages 9–30. American Mathematical Society, Providence, Rhode Island, 1973.
- [Ber75a] B. C. Berndt. Character analogues of the Poisson and Euler-MacLaurin summation formulas with applications. *Journal of Number Theory*, 7(4):413–445, November 1975.
- [Ber75b] B. C. Berndt. On Eisenstein series with characters and the values of Dirichlet L -functions. *Acta Arithmetica*, 28(3):299–320, 1975.
- [Ber76] B. C. Berndt. Dedekind sums and a paper of G. H. Hardy. *J. London Math. Soc.*, 13(2):129–137, May 1976.
- [Ber77] B. C. Berndt. Modular transformations and generalizations of several formulae of Ramanujan. *Rocky Mountain J. Math.*, 7(1):147–190, March 1977.
- [Ber89] B. C. Berndt. *Ramanujan’s Notebooks, Part II*. Springer-Verlag, New York, 1989.
- [Bol49] G. Bol. Invarianten linearer Differentialgleichungen. *Abh. Math. Semin. Univ. Hambg.*, 16:1–28, 1949.
- [CFI12] J. B. Conrey, D. W. Farmer, and Ö. Imamoglu. The nontrivial zeros of period polynomials of modular forms lie on the unit circle. *International Mathematics Research Notices*, August 2012.
- [CG14] P. Charollois and M. Greenberg. Rationality of secant zeta values. *Ann. Sci. Math. Quebec*, 2014. To appear.
- [Cho28] S. Chowla. Some infinite series, definite integrals and asymptotic expansions. *J. Indian Math. Soc.*, 17:261–288, 1927/28.

- [Coh22] A. Cohn. Über die Anzahl der Wurzeln einer algebraischen Gleichung in einem Kreise. *Mathematische Zeitschrift*, 14(1):110–148, December 1922.
- [EGR13] A. El-Guindy and W. Raji. Unimodularity of roots of period polynomials of Hecke eigenforms. *Preprint*, 2013.
- [GMR11] S. Gun, M. R. Murty, and P. Rath. Transcendental values of certain Eichler integrals. *Bull. Lond. Math. Soc.*, 43(5):939–952, May 2011.
- [Gro70] E. Grosswald. Die Werte der Riemannschen Zetafunktion an ungeraden Argumentstellen. *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II*, 1970:9–13, 1970.
- [Kat74] K. Katayama. Ramanujan’s formulas for L -functions. *Journal of the Mathematical Society of Japan*, 26(2):234–240, April 1974.
- [Lag92] J. L. Lagrange. Solution d’un problème d’arithmétique. In J.-A. Serret, editor, *Oeuvres de Lagrange*, volume 1, pages 671–731. Gauthier-Villars, Paris, 1867-1892.
- [Len02] H. W. Lenstra Jr. Solving the Pell equation. *Notices Amer. Math. Soc.*, 49(2):182–192, 2002.
- [LR13] M. N. Lalin and M. D. Rogers. Variations of the Ramanujan polynomials and remarks on $\zeta(2j+1)/\pi^{2j+1}$. *Functiones et Approximatio Commentarii Mathematici*, 48(1):91–111, March 2013.
- [LRR14] M. N. Lalin, F. Rodrigue, and M. D. Rogers. Secant zeta functions. *Journal of Mathematical Analysis and Applications*, 409(1):197–204, 2014.
- [LS13] M. N. Lalin and C. J. Smyth. Unimodularity of zeros of self-inversive polynomials. *Acta Mathematica Hungarica*, 138(1-2):85–101, 2013.
- [Miy06] T. Miyake. *Modular Forms*. Springer, 2006.
- [MSW11] M. R. Murty, C. J. Smyth, and R. J. Wang. Zeros of Ramanujan polynomials. *J. Ramanujan Math. Soc.*, 26(1):107–125, 2011.
- [PP13] V. Paşol and A. A. Popa. Modular forms and period polynomials. *Proc. Lond. Math. Soc.*, February 2013.
- [Raz77] M. J. Razar. Values of Dirichlet series at integers in the critical strip. In J.-P. Serre and D. B. Zagier, editors, *Modular Functions of One Variable VI*, number 627 in Lecture Notes in Mathematics, pages 1–10. Springer Berlin Heidelberg, January 1977.
- [Riv12] T. Rivoal. On the convergence of Diophantine Dirichlet series. *Proc. Edinb. Math. Soc.*, 55(02):513–541, February 2012.
- [Wei77] A. Weil. Remarks on Hecke’s lemma and its use. In S. Iyanaga, editor, *Algebraic Number Theory: Papers Contributed for the Kyoto International Symposium, 1976*, pages 267–274. Japan Society for the Promotion of Science, 1977.
- [Yos97] M. Yoshida. *Hypergeometric Functions, My Love*. Vieweg, 1997.
- [Zag91] D. Zagier. Periods of modular forms and Jacobi theta functions. *Inventiones mathematicae*, 104(1):449–465, December 1991.
- [ZR76] I. J. Zucker and M. M. Robertson. Some properties of Dirichlet L -series. *Journal of Physics A: Mathematical and General*, 9(8):1207–1214, August 1976.

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